Three Term Recurrence Relations for Szegő Type Polynomials

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Resumo: *We look at polynomials similar to the Szegő polynomials defined with respect to a linear functional $\mathcal{M}$ for which the moments $\mathcal{M}[t^n] = \mu_n$ are all complex and satisfy, either $\mu_{-n} = \overline{\mu}_n$ and $\Delta_n \neq 0$ or $\mu_{-n} = \mu_n$ and $\Delta_n \neq 0$, for $n \geq 0$. Here, Δ_n are the Toeplitz determinants associated with the moments μ_n. The main objective here is to characterize these polynomials form a three term recurrence relation.*

1 Introduction

Given the double sequence $\{\mu_n\}_{n=-\infty}^{\infty}$ of complex numbers, let the linear functional $\mathcal{M}$ on the space of Laurent polynomials be defined by

$$\mathcal{M}[t^m] = \mu_m, \quad m = 0, \pm 1, \pm 2, \ldots .$$

The functional $\mathcal{M}$ can be referred to as a moment functional.

Let Δ_n, $n = 0, 1, \ldots$, be the associated Toeplitz determinants as defined by:
$\Delta_0 = \mu_0$ and

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\ \vdots & \vdots & & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_1 \\ \mu_{-n} & \mu_{-n+1} & \cdots & \mu_0 \end{vmatrix}, \quad n \geq 1.$$

We consider the sequence of polynomials $\{S_n\}$ defined by

$$\mathcal{M}[t^{-m} S_n] = 0, \quad 0 \leq m \leq n-1, \ n \geq 1.$$

where S_n for any $n \geq 0$ is a monic polynomial of degree n.

If the moment functional $\mathcal{M}$ is such that $\Delta_n \neq 0$, $n \geq 0$, then it is easily seen that this sequence of polynomials exists uniquely. The following results can be easily established.

$$S_n(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\ \vdots & \vdots & & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_1 \\ 1 & z & \cdots & z^n \end{vmatrix}, \qquad (1.1)$$

for $n \geq 1$ and $\mathcal{M}[t^{-m} S_n] = \delta_{n,m} \Delta_n / \Delta_{n-1}$, for $0 \leq m \leq n$, $n \geq 1$.

If the moment functional $\mathcal{M}$ is such that $\mu_{-n} = \overline{\mu}_n$ for $n \geq 1$ and $\Delta_n > 0$ for $n \geq 0$, then the polynomials S_n are known as the Szegő polynomials (see for example, [3, 7]). In this case, $\mathcal{M}$ can be represented by a Stieltjes integral with respect to a positive measure on the unit circle. The recent publications of the books [4] and [5] by Barry Simon has shown the importance and the renewed interest in studying these polynomials. The particular case where all μ_n are real, thus $\mu_{-n} = \mu_n$ for $n \geq 1$, is of special interest because of the connection with orthogonal polynomials on the interval $[-1, 1]$. For a recent paper on this particular case, we refer to [1].

If the moment functional $\mathcal{M}$ is such that all μ_n are real, $\mu_{-n} = \mu_n$ for $n \geq 1$ and $(-1)^{n(n+1)/2} \Delta_n > 0$ for $n \geq 0$, then the associated polynomials S_n are also well known (see for example [2, 6, 8]). In this case, $\mathcal{M}$ can be represented by a Stieltjes integral associated with a positive measure on the positive (or negative) half of the real line.

The objective in this work is to consider the polynomials S_n when the moment functional $\mathcal{M}$ is subject to a less restricted condition than those assumed in [1, 2, 6, 8]. To be precise, we assume that $\mathcal{M}[t^n] = \mu_n$, $n = 0, \pm 1, \pm 2, \ldots$, are all complex and satisfy, either

$$\text{A:} \quad \mu_{-n} = \overline{\mu}_n \ \text{ and } \ \Delta_n \neq 0 \ \text{ for } \ n \geq 0 \quad (1.2)$$

or

$$\text{B:} \quad \mu_{-n} = \mu_n \ \text{ and } \ \Delta_n \neq 0 \ \text{ for } \ n \geq 0. \quad (1.3)$$

When the moments are real both of these conditions coincide.

Under these restrictions (1.2) or (1.3), we call $\{S_n\}$ the Szegő type polynomials associated with the moment functional $\mathcal{M}$. We have for these polynomials

$$\mathcal{M}[t^{-m} S_n] = \overline{\mathcal{M}[t^{-n+m} S_n^*]} = \delta_{n,m} \frac{\Delta_n}{\Delta_{n-1}}, \quad (1.4)$$

$0 \leq m \leq n$, $n \geq 1$, when (1.2) holds and

$$\mathcal{M}[t^{-m} S_n] = \mathcal{M}[t^{-n+m} S_n^\bullet] = \delta_{n,m} \frac{\Delta_n}{\Delta_{n-1}}, \quad (1.5)$$

$0 \leq m \leq n$, $n \geq 1$, when (1.3) holds.

Here, the symbol $*$ is used to represent the well known reciprocal polynomials defined by $P_n^*(z) = z^n \overline{P_n(1/\overline{z})}$ and the the symbol $\bullet$ is used to represent the inverted (or reversed) polynomials defined by $P^\bullet(z) = z^n P(1/z)$. In both cases, P_n is treated as a polynomial of degree n.

2 Some basic properties

Theorem 2.1 *Let $a_n = S_n(0)$ for $n \geq 1$, where S_n are the Szegő type polynomials associated with the moment functional $\mathcal{M}$. Then the following hold.*

A.1) The polynomials S_n satisfy the recurrence relations

$$\begin{aligned} S_n^*(z) &= \overline{a}_n z S_{n-1}(z) + S_{n-1}^*(z), \\ S_n(z) &= a_n S_n^*(z) + (1 - |a_n|^2) z S_{n-1}(z), \end{aligned} \quad n \geq 1;$$

A.2) $\quad |a_n| \neq 1$ *for all* $n \geq 1$,

when (1.2) *holds.*

B.1) The polynomials S_n satisfy the recurrence relations

$$\begin{aligned} S_n^\bullet(z) &= a_n z S_{n-1}(z) + S_{n-1}^\bullet(z), \\ S_n(z) &= a_n S_n^\bullet(z) + (1 - a_n^2) z S_{n-1}(z), \end{aligned} \quad n \geq 1;$$

B.2) $\quad a_n \neq \pm 1$ *for all* $n \geq 1$,

when (1.3) *holds.*

Proof. The proof of this theorem, especially of parts A.1 and A.2, are similar to analogous results in [3]. Thus we give only the proofs of parts B1 and B2.

For $n \geq 1$ set

$$C_n(z) = S_n^\bullet(z) - \gamma_n z S_{n-1}(z) - S_{n-1}^\bullet(z),$$

where $\gamma_n = -\mathcal{M}[t^{-n} S_{n-1}^\bullet] / \mathcal{M}[t^{-(n-1)} S_{n-1}]$.

We consider C_n as a polynomial of degree n and show that it is identically zero. Clearly,

$$\mathcal{M}[t^{-m} C_n] = 0 \ \text{ for } \ 1 \leq m \leq n. \quad (2.1)$$

Since $C_n(0) = 0$, we can write $C_n^\bullet(z) = \sum_{k=0}^{n-1} c_k S_k(z)$ and hence

$$C_n(z) = \sum_{k=0}^{n-1} c_k z^{n-k} S_k^\bullet(z).$$

Using here the results of (2.1) for $m = n$, $m = n - 1$ until $m = 1$ and noting at each stage that $\mathcal{M}[S_{n-m}^\bullet] \neq 0$, we successively obtain that $c_0 = 0$, $c_1 = 0$ until $c_{n-1} = 0$. Hence proving $C_n(z) \equiv 0$ and thus

$$S_n^\bullet(z) = \gamma_n z S_{n-1}(z) + S_{n-1}^\bullet(z).$$

Here, comparing the coefficients of z^n we obtain $\gamma_n = S_n(0)$, thus establishing the first of the recurrence relations in B1.

Now to prove the other recurrence relation, for $n \geq 1$ we set

$$C_n(z) = S_n(z) - a_n S_n^\bullet(z) - \gamma_n z S_{n-1}(z),$$

with $\gamma_n = \mathcal{M}[t^{-n} S_n] / \mathcal{M}[t^{-(n-1)} S_{n-1}]$. Hence,

$$\mathcal{M}[t^{-m} C_n] = 0 \ \text{ for } \ 1 \leq m \leq n.$$

Since $C_n(0) = 0$, we can follow the exact procedure as before and establish that $C_n(z) \equiv 0$ to obtain

$$S_n(z) = a_n S_n^\bullet(z) + \gamma_n z S_{n-1}(z).$$

Comparing the coefficients of z^n we obtain $\gamma_n = 1 - a_n^2$, thus establishing the second of the recurrence relations in B1.

Since $\mathcal{M}[t^{-n}S_n] = \Delta_n/\Delta_{n-1}$, we obtain from the above recurrence relation

$$1 - a_n^2 = \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}, \quad n \geq 1.$$

Here, $\Delta_{-1} = 1$. Hence from (1.3) we have $1 - a_n^2 \neq 0$, $n \geq 1$. Thus proving part B2 of the theorem. ■

3 Three term recurrence

If $a_n \neq 0$, $n \geq 1$, then from Theorem 2.1, the following three term recurrence relations can be established for the Szegő type polynomials S_n:

$$S_{n+1}(z) = \big(z + \beta_{n+1}\big)S_n(z) + \alpha_{n+1} z S_{n-1}(z),$$

$n \geq 1$, with $S_0 = 1$ and $S_1(z) = z + \beta_1$, where if (1.2) holds then $\beta_1 = a_1$ and, for $n \geq 1$,

$$\alpha_{n+1} = \beta_{n+1}(|a_n|^2 - 1), \ \beta_{n+1} = a_{n+1}/a_n \tag{3.1}$$

and if (1.3) holds then $\beta_1 = a_1$ and, for $n \geq 1$,

$$\alpha_{n+1} = \beta_{n+1}(a_n^2 - 1), \ \beta_{n+1} = a_{n+1}/a_n. \tag{3.2}$$

From (1.1), note that $a_n = (-1)^n \frac{\hat{\Delta}_{n-1}}{\Delta_{n-1}}$, where

$$\hat{\Delta}_{n-1} = \begin{vmatrix} \mu_1 & \cdots & \mu_n \\ \mu_0 & \cdots & \mu_{n-1} \\ \vdots & & \vdots \\ \mu_{-n+2} & \cdots & \mu_1 \end{vmatrix}, \quad n \geq 1.$$

Hence to establish the above three term recurrence relations, one must assume that the moment functional $\mathcal{M}$ is such that

$$\mu_{-n} = \overline{\mu}_n, \ \Delta_n \neq 0 \text{ and } \hat{\Delta}_n \neq 0, \ n \geq 0. \tag{3.3}$$

or

$$\mu_{-n} = \mu_n, \ \Delta_n \neq 0 \text{ and } \hat{\Delta}_n \neq 0, \ n \geq 0. \tag{3.4}$$

From (3.1) and (3.2), necessary conditions on the coefficients of the three term recurrence relation for S_n are

$$1 + \frac{\alpha_{n+1}}{\beta_{n+1}} = |a_n|^2 = \prod_{k=1}^{n} |\beta_k|^2 \quad \text{for} \quad n \geq 1,$$

when (3.3) holds and

$$1 + \frac{\alpha_{n+1}}{\beta_{n+1}} = a_n^2 = \prod_{k=1}^{n} \beta_k^2 \quad \text{for} \quad n \geq 1,$$

when (3.4) holds.

Now we show that any one of the the above two conditions is also sufficient for a sequence of polynomials $\{S_n\}$ satisfying the three term recurrence relation to be a sequence of Szegő type polynomials associated with some moment functional.

Theorem 3.1 *Let $\{S_n\}$ be the sequence of polynomials given by the recurrence relation*

$$S_{n+1}(z) = \big(z + \beta_{n+1}\big)S_n(z) + \alpha_{n+1} z S_{n-1}(z), \tag{3.5}$$

$n \geq 1$, with $S_0 = 1$ and $S_1(z) = z + \beta_1$, where $\alpha_{n+1} \neq 0$, $\beta_n \neq 0$ for $n \geq 1$.

A) If

$$1 + \frac{\alpha_{n+1}}{\beta_{n+1}} = |S_n(0)|^2 \quad \textit{for} \quad n \geq 1, \tag{3.6}$$

then there exists a moment functional $\mathcal{M}$, with its moments $\mu_m = \mathcal{M}[t^m]$, $m = 0, \pm1, \pm2, \ldots$ are such that (3.3) *holds and for which $\{S_n\}$ are the Szegő type polynomials satisfying* (1.4).

B) If

$$1 + \frac{\alpha_{n+1}}{\beta_{n+1}} = [S_n(0)]^2 \quad \textit{for} \quad n \geq 1, \tag{3.7}$$

then there exists a moment functional $\mathcal{M}$, with its moments $\mu_m = \mathcal{M}[t^m]$, $m = 0, \pm1, \pm2, \ldots$ are such that (3.4) *holds and for which $\{S_n\}$ are the Szegő type polynomials satisfying* (1.5).

Proof. We consider another sequence of polynomials $\{Q_n\}$ given by

$$Q_{n+1}(z) = \big(z + \beta_{n+1}\big)Q_n(z) + \alpha_{n+1} z Q_{n-1}(z),$$

$n \geq 1$, with $Q_0 = 0$ and $Q_1(z) = \mu_0\beta_1$, where μ_0 is an arbitrary non-zero real number.

Hence,

$$\frac{Q_1(z)}{S_1(z)} = \frac{\mu_0\beta_1}{S_1(z)} \quad \text{and}$$

$$\frac{Q_{n+1}(z)}{S_{n+1}(z)} - \frac{Q_n(z)}{S_n(z)} = \frac{Q_{n+1}(z)S_n(z) - S_{n+1}(z)Q_n(z)}{S_n(z)S_{n+1}(z)} = \frac{\mu_0\beta_1\alpha_2\cdots\alpha_{n+1}(-z)^n}{S_n(z)S_{n+1}(z)}, \qquad n \geq 1.$$

Hence, there exist two series expansions $L_0(z)=\sum_{i=0}^{\infty}\mu_{-i}z^i$ and $L_\infty(z)=\sum_{i=0}^{\infty}-\mu_{i+1}z^{-i-1}$, such that

$$\begin{aligned}
&L_0(z)-\frac{Q_n(z)}{S_n(z)}\\
&\quad=\frac{(-1)^n\mu_0\alpha_2\cdots\alpha_{n+1}}{\beta_1(\beta_2\cdots\beta_n)^2\beta_{n+1}}z^n+O(z^{n+1}),\\
&L_\infty(z)-\frac{Q_n(z)}{S_n(z)}\\
&\quad=(-1)^n\mu_0\beta_1\alpha_2\cdots\alpha_{n+1}\frac{1}{z^{n+1}}+O(\frac{1}{z^{n+2}}),
\end{aligned}\tag{3.8}$$

for $n\geq 1$. From these, setting $S_n(z)=s_0^{(n)}+s_1^{(n)}z+\ldots+s_n^{(n)}z^n$, we obtain $\mu_1=\tau_\infty^{(0)}=-\mu_0\beta_1$ and the set of equations

$$\begin{array}{rcrcl}
\mu_1 s_0^{(n)} & + \ \ldots\ + & \mu_{n+1}s_n^{(n)} & = & \tau_\infty^{(n)},\\
\mu_0 s_0^{(n)} & + \ \ldots\ + & \mu_n s_n^{(n)} & = & 0,\\
\vdots & & & & \vdots\\
\mu_{-n+1}s_0^{(n)} & + \ \ldots\ + & \mu_1 s_n^{(n)} & = & 0,\\
\mu_{-n}s_0^{(n)} & + \ \ldots\ + & \mu_0 s_n^{(n)} & = & \tau_0^{(n)},
\end{array}\tag{3.9}$$

for $n\geq 1$, where

$$\tau_\infty^{(n)}=(-1)^{n+1}\mu_0\beta_1\alpha_2\cdots\alpha_{n+1}$$
$$\tau_0^{(n)}=\frac{(-1)^n\mu_0\alpha_2\cdots\alpha_{n+1}}{\beta_2\cdots\beta_n\beta_{n+1}}.$$

Since $s_n^{(n)}=1$, using Cramer´s rule to these equations gives

$$\Delta_n=\tau_0^{(n)}\Delta_{n-1}\quad\text{and}\quad\hat{\Delta}_n=(-1)^n\tau_\infty^{(n)}\Delta_{n-1},$$

$n\geq 1$, where $\Delta_0=\mu_0$, $\hat{\Delta}_0=\mu_1=-\beta_1\Delta_0$ and

$$\Delta_n=\begin{vmatrix}
\mu_0 & \mu_1 & \cdots & \mu_n\\
\mu_{-1} & \mu_0 & \cdots & \mu_{n-1}\\
\vdots & \vdots & & \vdots\\
\mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_1\\
\mu_{-n} & \mu_{-n+1} & \cdots & \mu_0
\end{vmatrix},$$

$$\hat{\Delta}_n=\begin{vmatrix}
\mu_1 & \mu_2 & \cdots & \mu_{n+1}\\
\mu_0 & \mu_1 & \cdots & \mu_n\\
\vdots & \vdots & & \vdots\\
\mu_{-n+2} & \mu_{-n+3} & \cdots & \mu_2\\
\mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_1
\end{vmatrix}.$$

Therefore,

$$\Delta_n\neq 0,\quad\hat{\Delta}_n\neq 0,\quad n\geq 0.$$

If we define the moment functional $\mathcal{M}$ by

$$\mathcal{M}[t^m]=\mu_m,\quad m=0,\pm1,\pm2,\ldots,$$

then it is easily observed from (3.9) that $\mathcal{M}[tS_n(t)]=\tau_\infty^{(n)}$, $n\geq 0$ and

$$\mathcal{M}[t^{-m}S_n(t)]=\delta_{n,m}\tau_0^{(n)}=\delta_{n,m}\frac{\Delta_n}{\Delta_{n-1}},\tag{3.10}$$

$0\leq m\leq n$, $n\geq 1$. Furthermore, the coefficients of the three term recurrence relations satisfy, $\beta_1=-\mathcal{M}[t]/\mathcal{M}[1]$ and

$$\alpha_{n+1}=-\frac{\mathcal{M}[tS_n(t)]}{\mathcal{M}[tS_{n-1}(t)]},$$
$$\beta_{n+1}=-\alpha_{n+1}\frac{\mathcal{M}[t^{-n}S_n(t)]}{\mathcal{M}[t^{-(n-1)}S_{n-1}(t)]},\quad n\geq 1.$$

With (3.10), to complete the proof we need to show that $\mu_{-m}=\overline{\mu}_m$ or $\mu_{-m}=\mu_m$ for $m=1,2,\ldots$, depending on (3.6) or (3.7).

For this purpose we consider the polynomials D_n and C_n defined by

$$D_n(z)=\frac{\beta_{n+1}S_n(z)+\alpha_{n+1}zS_{n-1}(z)}{\beta_{n+1}+\alpha_{n+1}},$$
$$C_n(z)=\mathcal{M}\Big[\frac{D_n(z)-D_n(t)}{z-t}\Big],$$

for $n\geq 1$. Note that D_n is a monic polynomial of degree n. From the three term recurrence relation for the polynomials S_n we can also write

$$D_n(z)=\frac{S_{n+1}(z)-zS_n(z)}{\beta_{n+1}+\alpha_{n+1}},\quad n\geq 1.$$

From (3.10),

$$\mathcal{M}[t^{-m}D_n(t)]=\delta_{n,n-m}\frac{\alpha_{n+1}\tau_\infty^{(n-1)}}{\beta_{n+1}+\alpha_{n+1}},\tag{3.11}$$

$0\leq m\leq n$, $n\geq 1$ and thus $\check{\Delta}_0=\mu_{-1}\neq 0$ and

$$\check{\Delta}_n=\begin{vmatrix}
\mu_{-1} & \mu_0 & \cdots & \mu_{n-1}\\
\mu_{-2} & \mu_{-1} & \cdots & \mu_{n-2}\\
\vdots & & \vdots & \\
\mu_{-n-1} & \mu_{-n} & \cdots & \mu_{-1}
\end{vmatrix}\neq 0,\quad n\geq 1.$$

These relations also guarantee the existence of the following three term recurrence relations for D_n and C_n.

$$\begin{aligned}
D_{n+1}(z)&=(z+c_{n+1})D_n(z)+d_{n+1}zD_{n-1}(z),\\
C_{n+1}(z)&=(z+c_{n+1})C_n(z)+d_{n+1}zC_{n-1}(z),
\end{aligned}\tag{3.12}$$

$n \geq 1$, with $D_0(z) = 1$, $D_1(z) = z + c_1$, $C_0(z) = 0$ and $C_1(z) = \mu_0$, where $c_1 = D_1(0) = \frac{\beta_1\beta_2}{\beta_2+\alpha_2}$ and

$$d_{n+1} = -\frac{\mathcal{M}[D_n(t)]}{\mathcal{M}[D_{n-1}(t)]} = \frac{\beta_n + \alpha_n}{\beta_{n+1} + \alpha_{n+1}}\alpha_{n+1},$$
$$c_{n+1} = \frac{D_{n+1}(0)}{D_n(0)} = \frac{\beta_{n+1} + \alpha_{n+1}}{\beta_{n+2} + \alpha_{n+2}}\frac{S_{n+2}(0)}{S_{n+1}(0)} = \frac{\beta_{n+1} + \alpha_{n+1}}{\beta_{n+2} + \alpha_{n+2}}\beta_{n+2}, \tag{3.13}$$

$n \geq 1$. For example, the three term recurrence for D_n can be obtained by taking

$$D_{n+1}(z) - zD_n(z) = c_{n+1}D_n(z) + d_{n+1}zD_{n-1}(z) + P_{n-1}(z),$$

where d_{n+1} and c_{n+1} are such that $d_{n+1} = -\mathcal{M}[D_n(t)]/\mathcal{M}[D_{n-1}(t)]$ and P_{n-1} is a polynomial of degree less than or equal to $n-1$. Thus using the orthogonality (3.11) and the conditions $\check{\Delta}_n \neq 0$, $n \geq 0$ we can show that P_{n-1} is identically zero. Now the three term recurrence relation for C_n follows from the definition of C_n and $d_{n+1} = -\mathcal{M}[D_n(t)]/\mathcal{M}[D_{n-1}(t)]$.

Furthermore, from the definition of C_n,

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{\mu_i}{z^i} - \frac{zC_n(z)}{D_n(z)} &= O(\frac{1}{z^n}), \\ \sum_{i=0}^{\infty} -\mu_{-i-1}z^{i+1} - \frac{zC_n(z)}{D_n(z)} &= O(z^{n+1}), \end{aligned} \tag{3.14}$$

$n \geq 1$. Now using (3.6) we show that $\mu_{-n} = \overline{\mu}_n$, $n \geq 1$.

Firstly, (3.6) implies $\overline{\alpha}_n/\overline{\beta}_n = \alpha_n/\beta_n$, $n \geq 2$. From (3.14),

$$\begin{aligned} \sum_{i=0}^{\infty} \overline{\mu}_i z^i - \frac{\widetilde{C}_n(z)}{\widetilde{D}_n(z)} &= O(z^n), \\ \sum_{i=0}^{\infty} -\frac{\overline{\mu}_{-i-1}}{z^{i+1}} - \frac{\widetilde{C}_n(z)}{\widetilde{D}_n(z)} &= O(\frac{1}{z^{n+1}}), \end{aligned} \tag{3.15}$$

$n \geq 1$, where $\widetilde{D}_n$ and $\widetilde{C}_n$ are the monic polynomials given by

$$\begin{aligned} \widetilde{D}_n(z) &= (\bar{c}_1\bar{c}_2\cdots\bar{c}_n)^{-1}D_n^*(z) \\ \widetilde{C}_n(z) &= (\bar{c}_1\bar{c}_2\cdots\bar{c}_n)^{-1}C_n^*(z) \end{aligned} \quad n \geq 1.$$

Moreover, from the recurrence relations (3.12)

$$\begin{aligned} \widetilde{D}_{n+1}(z) &= (z + \frac{1}{\bar{c}_{n+1}})\widetilde{D}_n(z) + \frac{\bar{d}_{n+1}}{\bar{c}_n\bar{c}_{n+1}}z\widetilde{D}_{n-1}(z), \\ \widetilde{C}_{n+1}(z) &= (z + \frac{1}{\bar{c}_{n+1}})\widetilde{C}_n(z) + \frac{\bar{d}_{n+1}}{\bar{c}_n\bar{c}_{n+1}}z\widetilde{C}_{n-1}(z), \end{aligned} \tag{3.16}$$

$n \geq 1$, with $\widetilde{D}_0(z) = 1$, $\widetilde{D}_1(z) = z + \frac{1}{\bar{c}_1}$, $C_0(z) = 0$ and $C_1(z) = \frac{\mu_0}{\bar{c}_1}$. Thus, using (3.6) and (3.13), comparing the coefficients of the recurrence relations in (3.16) with the coefficients of the recurrence relations for S_n and Q_n, we obtain

$$\widetilde{D}_n(z) = S_n(z), \quad \widetilde{C}_n(z) = Q_n(z), \quad n \geq 1.$$

Hence, comparing the series expansions (3.8) and (3.15), we obtain $\mu_{-n} = \bar{\mu}_n$ for $n = 1, 2, \ldots$.

Analogously, with (3.7) we can prove that $\mu_{-n} = \mu_n$ for $n = 1, 2, \ldots$, by showing S_n and Q_n are, respectively, same as the polynomials

$$\begin{aligned} \widehat{D}_n(z) &= (c_1c_2\cdots c_n)^{-1}D_n^{\bullet}(z) \\ \widehat{C}_n(z) &= (c_1c_2\cdots c_n)^{-1}C_n^{\bullet}(z) \end{aligned} \quad n \geq 1.$$

This concludes the proof of the theorem. ■

Observe that when $\mu_{-n} = \overline{\mu}_n$, $n \geq 0$ then $\check{\Delta}_n = \overline{\hat{\Delta}}_n$, $n \geq 0$ and when $\mu_{-n} = \mu_n$, $n \geq 0$ then $\check{\Delta}_n = \hat{\Delta}_n$, $n \geq 0$.

4 Examples

1. Any choice of a sequence $\{a_n\}$ such that $0 < |a_n| < 1$, $n \geq 1$ and with this the coefficients of the three term recurrence relation (3.5) given by $\beta_1 = a_1$,

$$\alpha_{n+1} = \beta_{n+1}(|a_n|^2 - 1), \ \beta_{n+1} = a_{n+1}/a_n,$$

$n \geq 1$, then item A of the theorem 3.1 holds. It is well known that, in this case, $\mathcal{M}$ is a positive definite moment functional and that it can be represented by an integral with respect to a positive measure on the unit circle. The polynomials S_n are simply known as Szegő polynomials.

For exemple, with $\lambda > 0$, if we choose $a_n = \frac{\lambda}{n+\lambda}$, $n \geq 1$, then we obtain the polynomials $\{S_n\}$ given by the three term recurrence relation

$$S_{n+1}(z) = (z + \frac{n+\lambda}{n+\lambda+1})S_n(z) - \frac{n(n+2\lambda)}{(n+\lambda)(n+\lambda+1)}zS_{n-1}(z),$$

$n \geq 1$, with $S_0(z) = 1$ and $S_1(z) = z+\lambda/(\lambda+1)$. These are the Gegenbauer-Szegő polynomials orthogonal with respect to the moment functional $\mathcal{M}$ given by

$$\mathcal{M}[f] = \int_0^{2\pi} f(e^{i\theta})\,|\sin(\theta/2)|^{2\lambda} d\theta.$$

Note that the coefficients $a_n = \frac{\lambda}{n+\lambda}$, $n \geq 1$, being real imply item B of the theorem 3.1 also holds.

2. Given $\ell > 1$ let the sequence of positive numbers $\{\ell_n\}$ be given by

$$\ell_0 = 1 \quad \text{and} \quad \ell_n = \frac{(1+\ell)^n - (1-\ell)^n}{(1+\ell)^n + (1-\ell)^n}\ell, \ n \geq 1.$$

Let $\{S_n\}$ be the sequence of monic polynomials generated by the three term recurrence relation

$$S_{n+1}(z) = \big(z + \beta_{n+1}\big)S_n(z) + \alpha_{n+1} z S_{n-1}(z),$$

$n \geq 1$, with $S_0 = 1$ and $S_1(z) = z + \beta_1$, where

$$\beta_n = -\frac{\ell_n}{\ell_{n-1}}, \quad \alpha_{n+1} = -\frac{\ell_{n+1}}{\ell_n}(\ell_n^2 - 1), \quad n \geq 1.$$

Both item A and item B of the theorem 3.1 hold.

The polynomial S_n are the Szegő type polynomial associated with the moment functional $\mathcal{M}$ given by

$$\mathcal{M}[f] = \int_a^b f(t) \frac{1+t^{-1}}{\sqrt{b-t}\sqrt{t-a}}\, dt,$$

where $a^{-1} = b = (\ell + \sqrt{\ell^2 - 1})^2$.

3. Given $0 < q < 1$ let $\{S_n\}$ be the sequence of monic polynomials generated by the three term recurrence relation

$$S_{n+1}(z) = \big(z + \beta_{n+1}\big)S_n(z) + \alpha_{n+1} z S_{n-1}(z),$$

$n \geq 1$, with $S_0 = 1$ and $S_1(z) = z + \beta_1$, where

$$\beta_n = -q^{1/2}, \quad \alpha_{n+1} = -q^{1/2}(q^{-n} - 1), \quad n \geq 1.$$

Again, both item A and item B of the theorem 3.1 hold.

The polynomial S_n are the Szegő type polynomial associated with the moment functional $\mathcal{M}$ given by Moment functional

$$\mathcal{M}[f] = \int_0^\infty f(t)\; t^{-1} e^{\left(\frac{\ln(t)}{2\kappa}\right)^2} dt,$$

where $\kappa^2 = -\ln(\sqrt{q})$.

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