A combinatorial proof for an identity involving partitions with distinct odd parts

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Abstract: *We follow Santos et al [4] to present a combinatorial interpretation for a sum and provide a bijective proof for an identity involving the number of partitions of an integer in which odd parts are distinct and greater than 1.*

1 Introduction

In this article we present a bijective proof for the following identity

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{2n}}{(q^2;q^2)_n}=\prod_{n=0}^{\infty}\frac{1+q^{2n+3}}{1-q^{2n+2}}, \qquad (1)$$

where $(a;b)_n=(1-a)(1-ab)(1-ab^2)\cdots(1-ab^{n-1})$.

This identity can be obtained from the equation 2.2.7 (see Andrews [1], p. 19)

$$1+\sum_{n=1}^{\infty}\frac{(b-a)(b-aq)\cdots(b-aq^{n-1})z^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty}\frac{(1-azq^n)}{(1-bzq^n)},$$

by replacing q by q^2 and putting $a=-q, b=1$ and, finally, $z=q^2$:

$$1+\sum_{n=1}^{\infty}\frac{(1+q)(1+q^3)\cdots(1+q^{2n-1})q^{2n}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} = \prod_{n=0}^{\infty}\frac{(1+q^{2n+3})}{(1-q^{2n+2})},$$

what can be written as

$$1+\sum_{n=1}^{\infty}\frac{(-q;q^2)_n q^{2n}}{(q^2;q^2)_n}=\prod_{n=0}^{\infty}\frac{1+q^{2n+3}}{1-q^{2n+2}},$$

considering that $\frac{(-q;q^2)_n q^{2n}}{(q^2;q^2)_n}=1$ if $n=0$.

As in [4], we introduce parameters k and j in the LHS of (1) to obtain a combinatorial interpretation for the sum:

$$\sum_{n=0}^{\infty}\frac{(-q^k;q^{2k})_n q^{2nk}}{(q^{2j};q^{2k})_n}. \qquad (2)$$

The special case $k=j=1$ enable us to give a bijective proof for (1).

2 The combinatorial interpretation

In the theorem below we give a combinatorial interpretation for (2). In order to do that we define, for k and j positive integers, $A_{k,j}=\{ck+dj|c,d\geq 0\}$.

Theorem 1. *Let $f(1)=1$ and, for $n\geq 2$, $f(n)$ be the number of partitions of n into parts belonging to $A_{k,j}$ of the form $\lambda_1+\lambda_2+\cdots+\lambda_s$, $\lambda_t=c_tk+d_tj$, $c_s=2$ or 3, $d_t\equiv 0(mod\ 2)$ where for consecutive parts λ_t and λ_{t+1} we have*

- *if $c_t\equiv 0(mod\ 2)$ and $c_{t+1}\equiv 0(mod\ 2)$, then $c_t=c_{t+1}+d_{t+1}$;*
- *if $c_t\equiv 0(mod\ 2)$ and $c_{t+1}\equiv 1(mod\ 2)$, then $c_t=1+c_{t+1}+d_{t+1}$;*
- *if $c_t\equiv 1(mod\ 2)$ and $c_{t+1}\equiv 0(mod\ 2)$, then $c_t=1+c_{t+1}+d_{t+1}$;*
- *if $c_t\equiv 1(mod\ 2)$ and $c_{t+1}\equiv 1(mod\ 2)$, then $c_t=2+c_{t+1}+d_{t+1}$.*

Then

$$\sum_{n=0}^{\infty}f(n)q^n=\sum_{n=0}^{\infty}\frac{(-q^k;q^{2k})_n q^{2nk}}{(q^{2j};q^{2k})_n}.$$

Proof. Let $f(m,n)$ be the number of partitions enumerated by $f(n)$ having exactly m parts. Then the following recurrence relation is satisfaied by $f(m,n)$:

$$\begin{aligned} f(m,n) = \ & f(m,n-2km+2k-2j) \\ & +f(m-1,n-2k) \\ & +f(m-1,n-2km-k). \end{aligned} \tag{3}$$

To prove (3) we split the partitions enumerated by $f(m,n)$ into tree disjoint sets:

1. those for which $d_s \neq 0$;
2. those for which $2k$ is a part;
3. those for which $3k$ is a part.

- from those in set 1. we subtract $2j$ from the smallest part and $2k$ from the $m-1$ remaining parts. By doing this we are left with partitions of $n-2k(m-1)-2j$ in m parts that are enumerated by $f(m,n-2km+2k-2j)$;
- from those partitions in set 2. we subtract the part $2k$. Then, we obtain partitions of $n-2k$ in $m-1$ parts that are enumerated by $f(m-1,n-2k)$;
- from those in set 3. we subtract the part $3k$ and $2k$ from the other parts obtaining the partitions of $n-2k(m-1)-3k$ in $m-1$ parts that are enumerated by $f(m-1,n-2km-k)$.

Assuming that $f(0,n)=0$ if $n>0$, $f(0,0)=1$ and $f(l,n)=0$ if $l<0$, we define:

$$F(z,q)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m,n)z^m q^n.$$

By (3), we have
$F(z,q)=$
$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m,n-2km+2k-2j)z^m q^n$
$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m-1,n-2k)z^m q^n$
$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m-1,n-2km-k)z^m q^n$
$= q^{2j-2k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m,n-2km+2k-2j)q^{2km}z^m q^{n-2km+2k-2j} + zq^{2k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m-1,n-2k)z^{m-1}q^{n-2k} + zq^{3k}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m-1,n-2km-k)z^{m-1}q^{2k(m-1)}q^{n-2km-k}$
$= q^{2j-2k}F(zq^{2k},q) + zq^{2k}F(z,q) + zq^{3k}F(zq^{2k},q)$.

Let $h(m,q)=\sum_{n=0}^{\infty} f(m,n)q^n$, then $F(z,q)=\sum_{n=0}^{\infty} h(n,q)z^n$ and

$$\begin{aligned} \sum_{n=0}^{\infty} h(n,q)z^n &= q^{2j-2k}\sum_{n=0}^{\infty} h(n,q)(zq^{2k})^n \\ &+zq^{2k}\sum_{n=0}^{\infty} h(n,q)z^n \\ &+zq^{3k}\sum_{n=0}^{\infty} h(n,q)(zq^{2k})^n. \end{aligned}$$

Comparing the coefficient of z^n in the last equality we have

$$\begin{aligned} h(n,q) = \ & q^{2k(n-1)+2j}h(n,q)+q^{2k}h(n-1,q) \\ & +q^{2kn+k}h(n-1,q). \end{aligned}$$

Then

$$h(n,q)=\frac{q^{2k}(1+q^{2kn-k})h(n-1,q)}{1-q^{2k(n-1)+2j}}.$$

Observing that $h(0,q)=1$, we have

$$\begin{aligned} h(n,q) &= \frac{q^{2k}(1+q^{(2n-1)k})}{1-q^{2k(n-1)+2j}}\frac{q^{2k}(1+q^{(2n-3)k})}{1-q^{2k(n-2)+2j}}\cdots \\ &\cdots\frac{q^{2k}(1+q^{k})h(0,q)}{1-q^{2j}} = \frac{(-q^k;q^{2k})_n q^{2kn}}{(q^{2j};q^{2k})_n}. \end{aligned}$$

Therefore,

$$F(z,q)=\sum_{n=0}^{\infty} h(n,q)z^n=\sum_{n=0}^{\infty}\frac{(-q^k;q^{2k})_n q^{2kn}}{(q^{2j};q^{2k})_n}z^n.$$

To finish the proof we put $z=1$:

$$\begin{aligned} \sum_{n=0}^{\infty} f(n)q^n &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m,n)q^n \\ &= F(1,q) \\ &= \sum_{n=0}^{\infty}\frac{(-q^k;q^{2k})_n q^{2kn}}{(q^{2j};q^{2k})_n}. \end{aligned}$$

□

3 The combinatorial proof

It will be more apropriate for us to rewrite this theorem using a two-line notation as follows:

Theorem 2. *The generating function for matrices of the form*

$$\begin{pmatrix} c_1 \times k & c_2 \times k & \cdots & c_s \times k \\ d_1 \times j & d_2 \times j & \cdots & d_s \times j \end{pmatrix},$$

where c_t and d_t satisfy $c_s = 2$ or 3, $d_t \equiv 0(mod\ 2)$ and

- *if $c_t \equiv 0(mod\ 2)$ and $c_{t+1} \equiv 0(mod\ 2)$, then $c_t = c_{t+1} + d_{t+1}$;*
- *if $c_t \equiv 0(mod\ 2)$ and $c_{t+1} \equiv 1(mod\ 2)$, then $c_t = 1 + c_{t+1} + d_{t+1}$;*
- *if $c_t \equiv 1(mod\ 2)$ and $c_{t+1} \equiv 0(mod\ 2)$, then $c_t = 1 + c_{t+1} + d_{t+1}$;*
- *if $c_t \equiv 1(mod\ 2)$ and $c_{t+1} \equiv 1(mod\ 2)$, then $c_t = 2 + c_{t+1} + d_{t+1}$,*

is given by

$$\sum_{n=0}^{\infty} \frac{(-q^k;q^{2k})_n q^{2nk}}{(q^{2j};q^{2k})_n}.$$

Now, by taking $k = j = 1$ in this theorem and considering that the RHS of (1) is the generating function for partitions where the odd parts are distinct and greater than 1, we have the combinatorial interpretation for the identity (1) which is given in the next theorem. The proof we present is a bijective one.

Theorem 3. *The number of partitions of n where the odd parts are distinct and greater than 1 is equal to the number of matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix},$$

where c_t and d_t satisfy $c_s = 2$ or 3, $d_t \equiv 0(mod\ 2)$ and

- *if $c_t \equiv 0(mod\ 2)$ and $c_{t+1} \equiv 0(mod\ 2)$, then $c_t = c_{t+1} + d_{t+1}$;*
- *if $c_t \equiv 0(mod\ 2)$ and $c_{t+1} \equiv 1(mod\ 2)$, then $c_t = 1 + c_{t+1} + d_{t+1}$;*
- *if $c_t \equiv 1(mod\ 2)$ and $c_{t+1} \equiv 0(mod\ 2)$, then $c_t = 1 + c_{t+1} + d_{t+1}$;*
- *if $c_t \equiv 1(mod\ 2)$ and $c_{t+1} \equiv 1(mod\ 2)$, then $c_t = 2 + c_{t+1} + d_{t+1}$,*

with the sum of all entries equals to n.

Proof. In order to construct the bijection we associate to each two-line matrix

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix},$$

a partition $\lambda_1 + \cdots + \lambda_s$ by just adding the columns.

For example, the matrix $\begin{pmatrix} 3 & 2 & 2 \\ 2 & 0 & 0 \end{pmatrix}$ is associated with $5 + 2 + 2$.

We describe next how to go from a partition where the odd parts are distinct and greater than 1 to a two-line matrix.

1. we start at the and of the first line putting 2 or 3 if λ_s is even or odd, respectively. Then, we complete the column with an even number such that the sum of the entries of this column is equal to λ_s. So the last column is uniquely determined.
2. in order to create de second column, we must observe the parity of λ_{s-1} and λ_s because c_{s-1} and c_s have to satisfy the hypoteses above. For example, if c_{s-1} and c_s are both odd, then $c_{s-1} = 2+c_s+d_s$ and we complete this column by choosing an even number d_{s-1} such that $\lambda_{s-1} = c_{s-1}+d_{s-1}$. It is not diffiult to see that there is only one way to fill the column up.
3. to build the previous column we observe the parity of λ_{t-1} and λ_t following the procedure described in item 2.

Now we have to explain why by adding columns we get a partition where the odd parts are distinct and greater than 1. Observing how the two-line matrices are constructed we have that the entries of the second line are even numbers. For this reason the parity of each part obtained adding the entries in each column is given by the entry of the first line. So, it is impossible to have two consecutive odd parts $\lambda_t = c_t + d_t$ and $\lambda_t = c_{t+1} + d_{t+1}$ that are equal because we have $c_t = 2 + c_{t+1} + d_{t+1}$ in this case. As we begin with $c_s = 2$ or 3, each part obtained adding the entries in each column will be grater than 1. With this explanation we finished the proof.

□

We present, below, examples for $n = 9$ and $n = 10$ of the bijection described in Theorem 3.

The left column has the two-line matrices representations and in the right column the partitions in which the odd parts are distinct and greater than 1.

$\begin{pmatrix} 3 \\ 6 \end{pmatrix}$	9
$\begin{pmatrix} 3 & 2 \\ 4 & 0 \end{pmatrix}$	7+2
$\begin{pmatrix} 4 & 3 \\ 2 & 0 \end{pmatrix}$	6+3
$\begin{pmatrix} 5 & 2 \\ 0 & 2 \end{pmatrix}$	5+4
$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 0 & 0 \end{pmatrix}$	5+2+2
$\begin{pmatrix} 4 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$	4+3+2
$\begin{pmatrix} 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	3+2+2+2

Table 1: The matrices and partitions for $n = 9$

$\begin{pmatrix} 2 \\ 8 \end{pmatrix}$	10
$\begin{pmatrix} 2 & 2 \\ 6 & 0 \end{pmatrix}$	8+2
$\begin{pmatrix} 5 & 3 \\ 2 & 0 \end{pmatrix}$	7+3
$\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$	6+4
$\begin{pmatrix} 2 & 2 & 2 \\ 4 & 0 & 0 \end{pmatrix}$	6+2+2
$\begin{pmatrix} 5 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$	5+3+2
$\begin{pmatrix} 4 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$	4+4+2
$\begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix}$	4+2+2+2
$\begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	2+2+2+2+2

Table 2: The matrices and partitions for $n = 10$

In [3] Igor Pak asked for a combinatorial proof for the identity

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n^2} = \prod_{n=0}^{\infty} \frac{1+q^{2n+1}}{1-q^{2n+2}}. \tag{4}$$

This identity can be obtained from Corollary 2.4 in Andrews [1] which is

$$1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} \times \frac{(1-b)(1-bq)\cdots(1-bq^{n-1})(c/ab)^n}{(1-c)(1-cq)\cdots(1-cq^{n-1})}$$

$$= \prod_{m=0}^{\infty} \frac{(1-cq^m/a)(1-cq^m/b)}{(1-cq^m)(1-cq^m/ab)}$$

by first replacing $c = q^2$, $b = -q$, $q \to q^2$

$$1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq^2)\cdots(1-aq^{2n-2})}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} \times \frac{(1+q)(1+q^3)\cdots(1+q^{2n-1})(-q/a)^n}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}$$

$$= \prod_{m=0}^{\infty} \frac{(1-q^{2m+2}/a)(1+q^{2m+1})}{(1-q^{2m+2})(1+q^{2m+1}/a)}.$$

To get equation (4) all we have to do is to take $a \to \infty$.

Now in order to find a combinatorial interpretation fo the sum in (4) we introduce parameters in the following way

$$\sum_{n=0}^{\infty} \frac{(-q^k;q^{2^k})_n q^{kn^2}}{(q^{2j};q^{2k})_n (q^{2k};q^{2k})_n}. \tag{5}$$

Using the ideas described in [4] this paper it was possible to get the following interpretation for this sum: *the coeficient of q^n in (5) is the number of partition $\lambda_1 + \cdots + \lambda_s$ of n in which $\lambda_t = c_t k + d_t j$, $c_s \geq 1$, $d_t \equiv 0(mod\ 2)$ and*

- *if $c_t \equiv 0(mod\ 2)$ and $c_{t+1} \equiv 0(mod\ 2)$, then $c_t \geq 4 + c_{t+1} + d_{t+1}$;*
- *if $c_t \equiv 0(mod\ 2)$ and $c_{t+1} \equiv 1(mod\ 2)$, then $c_t \geq 3 + c_{t+1} + d_{t+1}$;*
- *if $c_t \equiv 1(mod\ 2)$ and $c_{t+1} \equiv 0(mod\ 2)$, then $c_t \geq 3 + c_{t+1} + d_{t+1}$;*
- *if $c_t \equiv 1(mod\ 2)$ and $c_{t+1} \equiv 1(mod\ 2)$, then $c_t \geq 2 + c_{t+1} + d_{t+1}$,*

We hope to be able to find a combinatorial proof for (4) using the ideas described in this paper.

References

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