Approach for stabilization by output feedback, invariant subspaces and Sylvester equations

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1 Introduction

A system is continuous time when all the variables of the system are known in every moment of time. A control system is called invariant time when its parameters are stationary with respect to time. The representation will be through systems of state equations.

The Sylvester equations play an important role in numerical linear algebra. For example, they arise in the computation of invariant subspaces, in control problems, as linearizations of algebraic Riccati equations, and in the discretization of partial differential equations. For small systems, direct methods are feasible.

This article deals with the problems of stabilization and of regional pole placement by static output feedback in linear continuous-time systems. The results are based on the key notion of (C,A,B)-invariant subspaces which can be characterized through a pair of coupled Sylvester equations. Then, solutions to these equations can be obtained, for systems verifying Kimura's condition $(m + p > n)$, into two steps using Syrmos and Lewis's algorithm.

The considered linear time-invariant system are described by :

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (1)$$
$$y(t) = Cx(t) \qquad (2)$$

where: $x \in \mathcal{X} \sim \Re^n$, $u \in \mathcal{U} \sim \Re^m$, $y \in \mathcal{Y} \sim \Re^p$.It is also assumed that B is full column-rank , C is full row-rank and that (C, A, B) is stabilizable and detectable. The studied problem is to find a static output feedback control law

$$u(t) = Gy(t) \quad , \quad G \in \Re^{m\times p} \qquad (3)$$

such that $\sigma(A + BGC) \in \mathcal{C}^-$, or equivalently, the closed-loop system is asymptotically stable. That is: , where $\sigma(A + BGC) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the set of eigenvalues of $A + BGC$.

This presents the basic concepts of theory of geometric control theory that base the approach adopted for the solution of the problem [11], [13]. The coupled Sylvester equation are interpreted on the basis of geometric control theory. Thus, through these equations, the stabilization problem using output feedback can be decomposed in two stages [9]: (i) determination of a (C, A)-outer detectable subspace, and (ii) inner stabilization of this subspace.

In the search of solutions for the stabilization problem they are argued and they presented procedures for the solution of these coupled equations, with emphasis in techniques of eigenstruture assignment [1] [9]. Thus, it is introduced notion of $O.S.$ (C, A, B)-invariant subspace and presented the coupled Sylvester equation. Thus too presented the algorithmic aspects, illustrating them with numerical examples. Some conclusive commentaries finally are presented.

2 Invariant Subspaces and coupled Sylvester Equations

As in [11], some concepts and definitions are used of geometric control theory [13]. It is know that an subespace $\mathcal{V} \subset \mathcal{X}$ is (A, B)-invariant if there exists $F : \mathcal{X} \longrightarrow \mathcal{Y}$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$, or equivalently , $A\mathcal{V} \subset \mathcal{V} + Im\, B$. For dually an subspace $\mathcal{T} \subset \mathcal{X}$ is (C, A)-invariant if exist $L : \mathcal{Y} \longrightarrow \mathcal{X}$ such that $(A + LC)\mathcal{T} \subset \mathcal{T}$, or equivalently, $\mathcal{T} \supset A(\mathcal{T} \cap Ker\,(C))$.

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Definition 2.1 *[11] An subspace $\mathcal{V} \subset \mathcal{X}$, of dimension v , is (C, A, B)-invariant if $\mathcal{V}$ is (A, B)-invariant and (C, A)-invariant.*

Consider $V \in \Re^{v \times v}$ such that $Im\,(V) = \mathcal{V}$ and consider $T \in \Re^{(n-v) \times v}$ an left annihilator of V, *i.e.* : $Ker\,(T) = Im\,(V)$. The definition 2.1 is equivalent an existence of matrices ($H_V \in \Re^{v \times v}$, $W \in \Re^{m \times v}$) and ($H_T \in \Re^{n-v \times n-v}$, $U \in \Re^{(n-v) \times v}$), solutions for the following *coupled Sylvester Equations*:

$$\begin{aligned} AV - VH_V &= -BW & (4)\\ TA - H_T T &= -UC & (5)\\ TV &= 0 & (6) \end{aligned}$$

An definition 2.1 and the equations (4) , (5) and (6) they have a basic importance in the treatment of the control problem using static output feedback, mainly for Eigenstructure Assignment [9] [11]. The present study it takes in account the properties of stabilizability and detectability, through the two definition to follow:
(i) an subspace (A, B)-invariant $\mathcal{V}$ is (A, B)-inner stabilizable subspace, that is there exist F such that $(A + BF)|\mathcal{V}$ is asymptotically stable; and
(ii) an subspace (C, A)-invariant $\mathcal{V}$ is (C, A)-outer detectable subspace if there exist L such that $(A + LC)|\mathcal{X}/\mathcal{V}$ is asymtotically stable.

Definition 2.2 *[11] An subspace $\mathcal{V}$, of dimension v, is (C, A, B)-invariant output stabilizable subspace , (or simply O.S. (C, A, B)-invariant [1]) if $\mathcal{V}$ is (A, B)-inner stabilizable and (C, A)-outer detectable.*

Thus, a necessary and sufficient condition for the existence $\mathcal{V} = Im(V)$ subspace either *O.S.* (C, A, B)-invariant subspace is that (4), (5) and (6) they are verified with the conditions additional of stability :

$$\begin{aligned} \sigma(H_V) \in \mathcal{C}^- & \quad (7)\\ \sigma(H_T) \in \mathcal{C}^- & \quad (8) \end{aligned}$$

A Theorem to follow it relates the concept of *O.S.* (C, A, B)-invariant subspace to the existence of a law of control of the type static output feedback (3) that it stabilizes the system in closed loop.

Theorem 2.1 *There exists an output feedback matrix $G : \mathcal{Y} \longrightarrow \mathcal{U}$ such that $\sigma(A + BGC) \in \mathcal{C}^-$, if and only if the following conditions are verified for some matrices ($V \in \Re^{n \times v}, H_V \in \Re^{v \times v}, W \in \Re^{m \times v}$), ($T \in \Re^{n-v \times n}, H_T \in \Re^{n-v \times n-v}, U \in \Re^{n-v \times v}$) to some scale positive $v \leq n$:*

$$\begin{aligned} AV - VH_V &= -BW, \textit{with } \sigma(H_V) \in \mathcal{C}^- & (9)\\ TA - H_T T &= -UC, \textit{with } \sigma(H_T) \in \mathcal{C}^- & (10)\\ TV &= 0 & (11)\\ Ker\,(CV) &\subseteq Ker\,(W) & (12)\\ Ker\,(B'T') &\subseteq Ker\,(U') & (13) \end{aligned}$$

where: $rank\,(V) = v$ e $rank\,(T) = n - v$. $\diamond$

[1] Output Stabilizable (C,A,B)-invariant [11] [3]

It is important to notice that these results have been presented and explored under different forms in the literature related to the eigenstruture assignment by output feebaback (see [12]). The presented statement corresponds to the statement of the Theorem 3.2 de [11]. As it was seen, the *coupled Sylvester Equations* (9), (10) e (11) describe some geometric properties of subspace $\mathcal{V} = Im\,(V)$. Under the restriction of imposed stability a matrix H_V, the equation (9) means that the subspace $\mathcal{V} = Im\,(V)$ must be (A, B)-inner stabilizable subspace; In a dual manner, (10) means that the subspace $Ker\,(T) = \mathcal{V}$ must be (C, A)-outer detectable subspace. Thus, under the coupling condition (11), is had that the existence of $\mathcal{V} = Ker\,(T)$ *O.S.* (C, A, B)-invariant subspace,with the conditions you add (12) and (13) is a necessary and suficient condition for the existence solution of the considered problem. Also notices that the conditions (12) and (13) correspond to the existence of a matrix $G \in \Re^{m \times p}$ that it verifies the two following equalities:

$$\begin{aligned} GCV = W & \quad (14)\\ TBG = U & \quad (15) \end{aligned}$$

Is important to observe that guaranteed the existence of solution for the Sylvester equations and that G has been gotten such that (14) (or 15) either verified, the eigenvalues in closed loop are given by:

$$\sigma(A + BGC) = \sigma(H_V) \,\dot{\cup}\, \sigma(H_T)$$

3 Algorithmic aspects

The established matrices conditions in the Theorem 2.1 are nonlinear in relation to the matrices variable (V, H_V, W, T, H_T, U). In particular, the terms nonlinear VH_V and $H_T T$ presented in (9) and (10), respectively, can be linear when fixing the matrices H_V and H_T, or equivalent when defining itself the eigenvalues desired for the system in closed loop.

In this section the basic procedure based in two algorithmic techniques of eigenstructure assignment for the attainment a output feedback are presented. The first technique, the algoritm proposed for Syrmos and Lewis in [9] solves the coupled Sylvester equations in two steps and it can directly be applied the systems that verify the condition $m + p > n$, known as condition of Kimura [8]. The second technique, the algorithm proposed for Paraskevopoulos in [1], uses a representation of the system in an adequate base of the state space, to rewrite the coupled Sylvester equations in the form of a set bilinear algebraic equations. In the two techniques, it is assumed that the dimension of $\mathcal{V} = Im\,(V)$ is equal to the number of output p, looking guaranteeing that a matrix $CV \in \Re^{p\times p}$ either invertible and, therefore, that the gain feedback matrix can be found, from (14), under the form $G = W(CV)^{-1}$.

In the sequence, it is considered system represented (C, A, B) is controlable and observable, is this necessary condition for the arbitrary choice of poles to be located in closed loop. This condition can be however to be relaxed for stabilizable and detectabilizable of (C, A, B), what it implies in the use of the non-controlable eigenvalues and non-observable in the spectrum desired for the system in closed loop.

In both the techniques to be presented, are considered that the set of poles desired is denoted by $\Lambda = \{\lambda_1, \ldots, \lambda_{n-p}, \lambda_{n-p+1}, \ldots, \lambda_n\} = \{\Lambda_T, \Lambda_V\}$ where Λ_T and Λ_V are auto-conjugated. These sets are associated the matrices $H_V \in \Re^{p\times p}$ and $H_T \in \Re^{(n-p)\times(n-p)}$ in the following form: $\sigma(H_T) = \Lambda_T = \{\lambda_1, \ldots, \lambda_{n-p}\}$ e $\sigma(H_V) = \Lambda_V = \{\lambda_{n-p+1}, \ldots, \lambda_n\}$. For simplicity, it is considered that the elements of $\Lambda = \sigma(A+BGC)$ are distinct.

3.1 Modified algorithm of Syrmos and Lewis

Based in the express conditions through Theorem 2.1, the following algorithm, considered in [9], generally it leads to a output feedback matrix that stabilizes the system in closed loop when the Kimura's condition $n < m + p$ is verified:

Step 1: It chooses the matrix $H_T \in \Re^{n-p\times n-p}$ such that $\sigma(H_T) = \Lambda_T \in \mathcal{C}^-$ and solve the Sylvester equation (10) for find an matrix $T \in \Re^{n-p\times n}$ such that

$$rank\left(\begin{bmatrix} T \\ C \end{bmatrix}\right) = n \Longleftrightarrow Ker\ T \cap Ker\ C = \{0\} \quad (16)$$

Step 2: Solve the Sylvester equation (9), for some matrix $H_V \in \Re^{p\times p}$ such that $\sigma(H_V) = \Lambda_V \in \mathcal{C}^-$ leading in consideration that the matrix V must verify the coupling condition (11) and that $rank\,(V)$ must be equal p.
Step 3: For construction, the relation (16) guarantees that the rank $(CV) = p$ and the matrix G can be calculated as only solution of (14).
∘

Is presented to follow it some comments with the double objective to show relative aspects to the numerical solution of steps algorithm and to justify them theoretically.
Remark 1.1 : The steps 1 and 2 can be resolved using techniques standards for eigenstructure assignment. Considering the matrices H_T e H_V in the diagonal form of Jordan, the following procedure can be adopted:

Step 1: Find $t_j \in \mathcal{C}^n$ and $u_j \in \mathcal{C}^p$, such that:

$$\begin{bmatrix} t_j' & u_j' \end{bmatrix}\begin{bmatrix} A - \lambda_j I \\ C \end{bmatrix} = 0\ \forall\ j = 1, \ldots, n-p \quad (17)$$

The row matrix $T \in \Re^{(n-p)\times n}$, denoted for T_j, are formed from the vectors t_j, as follows :

- if $\lambda_j \in \Re$, then $T_j = t_j'$;
- if $\lambda_j \in \mathcal{C}$, is considered $\lambda_{j+1} = \lambda_j^*$ and $\begin{cases} T_j = Re\,(t_j') \\ T_{j+1} = Imag\,(t_j') \end{cases}$.

Step 2: To determine $v_i \in \mathcal{C}^n$ and $w_i \in \mathcal{C}^n$ such that:

$$\begin{bmatrix} A - \lambda_i I & B \\ T & 0 \end{bmatrix}\begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0\ \forall\ i = n-p+1, \ldots, n \quad (18)$$

In similar form to the previous case, the used matrices V and W for the calculation of K can only be built with real elements. In particular: if $\lambda_i \in \mathcal{C}$, is considered $\lambda_{i+1} = \lambda_i^*$ and $\begin{cases} V_i = Re\,(v_i), & V_{i+1} = Imag\,(v_i) \\ W_i = Re\,(w_i), & W_{i+1} = Imag\,(w_i) \end{cases}$, where v_i and w_i denote the columns of the matrices V and W, respectively .
Remark 1.2: [4] In the step 1, under the observability (detectability) condition it is always possible to build to a matrix T that verifies condition (17) and that, therefore, it guarantees the obtaining of K saw (14). Once certain the matrix T, the step 2 is associated with the joint solution of the equations (9) and (11), or either to (A,B)-invariance of $Ker\,(T)$. This property is associated then to the "zero equation" (18) [2]. To guarantee that the eigenvalues λ_i of step 2 are freely assignable, the system matrix $P(\lambda) = \begin{bmatrix} A - \lambda I & B \\ T & 0 \end{bmatrix}$ of dimension $(2n-p)\times(n+m)$, must have complete rank for row $\forall\lambda$. The algorithm above is based in the fact that under the condition $m+p > n$, the system matrix $P(\lambda)$ it does not have loss of rank for almost all the triples (A, B, T) [10]. Thus the zero equation (18) has solutions for all λ_i. In the cases where it will have the presence of invariant zeros is possible to remake the Step 1 on the search of other solutions for the Sylvester equation (9).
Remark 1.3: In the less restrictive case

$m + p = n$, the system $P(\lambda)$ of dimension $(2n-p)\times(2n-p)$, is an square matrix and almost all the triples (A, B, T) have p invariant zeros finite stable [9]. In this case, the basic procedure can produce to a output feedback matrix that stabilizes the system in closed loop only if the matrix T, found in step 1, generate p stable invariant zeros which should be used to solve the equation (18) [2].

Example 3.1 : *Consider an linear system (1), (2), defined for [7]:*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The corresponding system is controllable and observable and $m + p = 5 > n$. The eigenvalues to assignment are given for : $\Lambda_T = \{-4\} \cup \Lambda_V = \{-3, -2+2j, -2-2j\}$.

Step 1 : *For $\lambda_1 = -4$, determined T that verifies (17) and such that (A, B, T) not have zeros invariant :*

$$T = [\ -0.3148 \quad 0.0630 \quad -0.2406 \quad -0.2425\ ];$$

Step 2 : *For $\lambda_2 = -3$, $\lambda_3 = -2+2j$ and $\lambda_4 = -2-2j$, determined V and W using (18) :*

$$V = \begin{bmatrix} 0.0727 & 0.0564 & 0.0594 \\ -0.2182 & -0.2315 & -0.0061 \\ 0.0242 & -0.0008 & 0.0289 \\ -0.1751 & -0.1325 & -0.1074 \end{bmatrix},$$

$$W = \begin{bmatrix} 0.8000 & 0.6504 & -0.5041 \\ 0.5252 & 0.4797 & -0.0502 \end{bmatrix}$$

Step 3 : *Determined G such that $GCV = W$:*

$$G = \begin{bmatrix} -61.0000 & -6.5685 & -30.8197 \\ -19.7741 & -5.6724 & -12.0000 \end{bmatrix}$$

The matrix in loop closed $A_G = A + BGC$ correspondent, whose the eigenvalues are the desired, is :

$$A_G = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ -60.0000 & 1.0000 & -6.5685 & -30.8197 \\ -1.0000 & 0.0000 & 0.0000 & 0.0000 \\ -19.7741 & 0.0000 & -5.6724 & -12.0000 \end{bmatrix}.$$

△

3.2 Algorithm of Paraskevopoulos

The presented technique to follow is based on the use of a transformation of coordinates, gotten from a decomposition of the matrix C, that it allows to solve the equation (17) saw a system auxiliary of reduced order $n - p$.

This type of decomposition is used in the theory of control mainly for the construction of observers of minimum order [1] [5]. Geometric point of view, it it aims at the construction subspace $\mathcal{V} = Ker\,(T)$ outer stabilizable and the guarantee of the relation $Ker\,(T) \cap Ker\,(C) = \{0\}$.

Consider a base change given for

$$x = [\ P_1 \quad P_2\ ] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (19)$$
$$P_1 \in \Re^{n\times p}, \quad P_2 \in \Re^{n\times n-p}$$

where $P = [\ P_1 \quad P_2\ ]$ is not singular matrix such that:

$$C\,[\ P_1 \quad P_2\ ] = [\ C_1 \quad 0\ ]\ , \qquad (20)$$
$$\text{with } C_1 \in \Re^{p\times p} \text{ and } rank(C_1) = p$$

The inverse of the matrix P é denoted for $\bar{P} = \begin{bmatrix} \bar{P}_1' \\ \bar{P}_2' \end{bmatrix}$, with $\bar{P}_1 \in \Re^{n\times p}$, $\bar{P}_2 \in \Re^{n\times n-p}$: $[\ P_1 \quad P_2\ ] \begin{bmatrix} \bar{P}_1' \\ \bar{P}_2' \end{bmatrix} = I_n$.

In this base, the system in open loop takes the form

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \\ y(t) &= [\ C_1 \quad 0\ ] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned} \qquad (21)$$

where, in particular, $A_{12} \in \Re^{p\times n-p}$ and $A_{22} \in \Re^{n-p\times n-p}$.

Lemma 3.1 *If (C, A) is observable, then always exists $T \in \Re^{n-p\times p}$ such that the equation (10) is verified and $Ker\,(T) \cap Ker\,(C) = \{0\}$.*

Proof:
As shown in [5], the observability of the pair (C, A) is equivalent to the observability of the pair (A_{12}, A_{22}) defined from the change base express in (19). They are the matrices $T_2 \in \Re^{(n-p)\times(n-p)}$, with *posto* $(T_2) = n - p$, and $T_1 \in \Re^{(n-p)\times p}$, solutions of Sylvester equations of *order-reduced*

$$T_2 A_{22} - H_T T_2 = -T_1 A_{12}\ , \quad \sigma(H_T) \in \mathcal{C}^- \qquad (22)$$

where the matrix $H_T \in \Re^{n-p\times n-p}$ is chosen such that $\sigma(H_T) \in \mathcal{C}^-$. Is important to observe that generally it is possible to find T_2 invertible when H_T does not contain the eigenvalues of the matrix A_{22} [1]. Can then, to calculate, $U \in \Re^{n-p\times p}$ as the only solution of:

$$UC_1 = -\left(T_1 A_{11} + T_2 A_{21} - H_T T_1\right) \qquad (23)$$

Jointly, (22) and (23) verify:

$$[\ T_1 \quad T_2\ ]\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - H_T[\ T_1 \quad T_2\ ] = U[\ C_1 \quad 0\ ]$$

that, in the original base, it corresponds to Sylvester equation (10). Moreover, as T_2 is invertible, is had:

$$rank\left(\begin{bmatrix} T \\ C \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ C_1 & 0 \end{bmatrix}\begin{bmatrix} \bar{P}'_1 \\ \bar{P}'_2 \end{bmatrix}\right) = n \iff Ker\,(T) \cap Ker\,(C) = \{0\}.$$

$\diamond$

The equations (10) and (22) are related to the construction of observers of minimum order [5] [1].

It can, in particular, to associate the eigenvalues of H_T to the eigenvalues of $(A_{22}+L_2A_{12})$ where $L_2 \in \Re^{n-p\times p}$ is such that

$$T_2L_2 \;=\; T_1 \tag{24}$$

Thus, substituting (24) in (22) is had

$$T_2(A_{22}+L_2A_{12}) \;=\; H_TT_2 \tag{25}$$

or either

$$\sigma(H_T) = \sigma(A_{22}+L_2A_{12}) \in \mathcal{C}$$

Is important to observe that the decomposition P used it is not only. In particular, it can computational be gotten through ortogonal transformations, $\bar{P} = P'$ [1] [3].

The Lemma 3.1 can, in particular, to be used to carry through the step 1 Syrmos and Lewis algorithm, presented in the previous section. In this section, it will be used to enunciate the Theorem 2.1 in presented alternative form originally for Paraskevopoulos algorithm in [1].

Theorem 3.1 *Given the system (1), (2),considers a decomposition in the form (21). Either G the output feedback express for* $G = W(CV)^{-1}$. *Then the matrix G verifies* $\sigma(A+BGC) = \sigma(H_V)\,\dot{\cup}\,\sigma(H_T)$, *if and only if*

$$[\ T_1 \quad T_2\ ]\begin{bmatrix} \bar{P}'_1 \\ \bar{P}'_2 \end{bmatrix}V \;=\; 0 \tag{26}$$

where the matrices ($W \in \Re^{m\times p}, V \in \Re^{n\times p}$) *and* ($T_1 \in \Re^{n-p\times p}, T_2 \in \Re^{n-p\times n-p}$) *satisfy the Sylvester equation (9) and (22),respectively, with* $det\,(CV) \neq 0$.

From the formularization presented in the Theorem 3.1 above, the following procedure is suggested in [1] to get poles assignment when $mp \geq n$.

3.2.1 Procedure for the determination of output feedback matrix

Considering that H_V and H_T are matrices in the diagonal form of Jordan, from (9) can be written W e V in the form

$$W = [\ w_1, \quad w_2, \quad \ldots, \quad w_p\ ]; \tag{27}$$

$$V = [\ X^1w_1, \quad X^2w_2, \quad \ldots, \quad X^pw_p\ ] \tag{28}$$

where

$$X^i = (\lambda_iI - A)^{-1}B, \quad for \quad i = 1, \ldots, p \tag{29}$$

In the same way, from (22) the matrices T_1 and T_2 have the form

$$T_1 = \begin{bmatrix} \phi'_{p+1} \\ \phi'_{p+2} \\ \vdots \\ \phi'_n \end{bmatrix} \quad and \quad T_2 = \begin{bmatrix} \psi'_{p+1} \\ \psi'_{p+2} \\ \vdots \\ \psi'_n \end{bmatrix}$$

where ψ_j , $\forall j = p+1, \ldots, n$, can be gotten of:

$$\psi'_j = \phi' A_{12}(\lambda_jI - A_{22})^{-1} \tag{30}$$

From the notation above, the coupling equation (26) can be written in the form of a bilinear system algebraic equations. Thus, substituting (29) and (30) in (26), is had:

$$\phi'_j\ [\ I_p \quad A_{12}[\lambda_jI - A_{22}]^{-1}\ ]\begin{bmatrix} \bar{P}'_1 \\ \bar{P}'_2 \end{bmatrix}X^iw_i = 0 \tag{31}$$

$$for \quad i = 1, \ldots, p \quad and \quad j = p+1, \ldots, n$$

Solving the equation (31), the parameters are determined for w_i and ϕ'_j. Then, the matrices W and V are defined from the equation (27)and the output feedback matrix G is directly calculated of (14).

From this formularization, the justification to follow is given in [1] for the obtainment of condition less restrictive $mp \geq n$ for arbitrary poles positioning for output feedback: "The equation (27)provides $p(m-1)$ free parameters (from the arbitrary elements of $w_1, w_2, \ldots, w_p$). Additionally the equation (30) provides $(p-1)(n-p)$ free parameters (from the arbitrary elements of $\phi_{p+1}, \phi_{p+2}, \ldots, \phi_n$). On the other hand the equation (31) results in a system of $p(n-p)$ independent equations. Thus for the solution existence, the number of equations must be equal or smaller that the addition of the free parameters. After algebraic manipulations it is gotten condition $mp \geq n$" .

Example 3.2 *: To follow, the procedure of Paraskevopoulos is applied to get the poles assignment for the system defined for A and B, used in the previous example. Is considered in this that only* 2 *states are measurable or either:*

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The corresponding system is controllable and observable being $mp = 4 = n$, *com* $p = 2$ *e* $m = 2$. *The eigenvalues to assignment are* $\Lambda_V = \{-1, -2\}$ *and* $\Lambda_T = \{-3, -4\}$.

In first place it finds a decomposition

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

using (21) such that:

$$C\begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} \bar{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and is calculated

$$\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \begin{bmatrix} \bar{P}_1' \\ \bar{P}_2' \end{bmatrix} AP_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

For $\lambda_1 = -1, \lambda_2 = -2$,*we fix the 2 free parameters in* $\begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} 1.0 & 1.0 \\ w_{12} & w_{22} \end{bmatrix}$,*to get* $w_{12} = -2.2$ *and* $w_{22} = 2.8$. *On the other hand for* $\lambda_3 = -3, \lambda_4 = -4$, *we fix the 2 free parameters in* $\begin{bmatrix} \phi_3' \\ \phi_4' \end{bmatrix} = \begin{bmatrix} 1.0000 & \phi_{32} \\ 1.0000 & \phi_{42} \end{bmatrix}$, *to get* $\phi_{32} = -1.6364$ *and* $\phi_{42} = -1.8824$. *In this form the matrices are determined* (W, V) *and* (T_1, T_2, T) *that satisfies the Sylvester equations (9), (22) and (31),respectively with* $det(CV) \neq 0$:

$$V = \begin{bmatrix} 0.6 & -0.8 \\ -1.6 & 0.6 \\ 1.6 & -1.2 \\ 0.6 & -0.4 \end{bmatrix} \quad ; \quad W = \begin{bmatrix} 1.0 & 1.0 \\ -2.2 & 2.8 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1.0000 & -1.6364 \\ 1.0000 & -1.8824 \end{bmatrix}; T_2 = \begin{bmatrix} -0.5455 & 0.1818 \\ -0.4706 & 0.1176 \end{bmatrix}$$

$$T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \bar{P}_1' \\ \bar{P}_2' \end{bmatrix}$$

$$T = \begin{bmatrix} -0.5455 & 0.1818 & 1.0000 & -1.6364 \\ -0.4706 & 0.1176 & 1.0000 & -1.8824 \end{bmatrix}$$

The system (A, B, T) *has as invariant zeros and stable eigenvalues:* $\{-1, -2\}$. *The output feedback matrix correspondent* G *is determined by* $GCV = W$:

$$G = \begin{bmatrix} -12.5 & 35.0 \\ -10.0 & 23.0 \end{bmatrix}$$

The matrix in closed loop $A_G = A + BGC$ *correspondent,whose the eigenvalues are the desired , it is:*

$$A_G = \begin{bmatrix} 0.0 & 1.0 & -12.5 & 35.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & -10.0 & 24.0 \\ -1.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}.$$

$\triangle$

4 Conclusion

Some base results had been presented in this article that basis the algorithms and theoretical results. The concept of subspace was presented *O.S.* (C, A, B)-invariant e its relation with the solution of stabilization problem using output feedback , by two coupled Sylvester equations.

Is distinguished the fact of the algorithm of Syrmos modified to allow to the solution of the problem in 2 sequence steps, for systems that verify the condition of Kimura $(m + p > n)$. As the algorithm, is based on the solution of a system of bilinear equations for the attainment of poles assignment, having been able to be applied the systems that verify the condition less restrictive $mp \geq n$.

The character of solution in two steps of the first algorithm still will be explored in the searche future. It is important to point out that this procedure also can solve the stabilization problem in systems that do not verify the Kimura condition, using itself compensating dynamic of order $\nu > n - (m + p)$.

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