Nonconvex Optimal Control Problems with Nonsmooth Mixed State Constraints

Maria do Rosário de Pinho,
ISR and DEEC, Faculdade de Engenharia da Universidade do Porto,
Rua Dr. Roberto Frias , 4200-465 Porto,Portugal
Email: mrpinho@fe.up.pt

Geraldo Nunes Silva
Depto de Ciências de Computação e Estatística, IBILCE, UNESP,
15054-000, São José do Rio Preto, SP
E-mail: gsilva@ibilce.unesp.br

Abstract: *Necessary conditions of optimality in the form of a weak maximum principle are derived for optimal control problems with mixed constraints. Such conditions differ from previous work since they hold when a certain convexity assumption is replaced by an "interiority" assumption. Notably the result holds for problems with possibly nonsmooth mixed constraints and with additional pointwise set control constraints. Essential to all the analysis is a nonsmooth version of the well known positive linear independence conditions on the mixed constraints.*

1 Introduction

Here we focus on necessary conditions for optimal control problems with nonsmooth mixed constraints. Necessary conditions in the form of maximum principles for problems with smooth mixed constraints have been addressed by a number of authors; see for example [9], [13], [14], [8], [11], to name but a few. However derivation of necessary conditions covering problems with nonsmooth mixed constraints remains a largely unexplored area (an exception may be found in [10] where autonomous problems are considered), a surprising fact taking into account the fast development of nonsmooth methods for optimal control since the publication of the seminal book [1].

Although a weak maximum principle for optimal control problems with nonsmooth inequality mixed constraints is derived in [6], such result holds under a convexity hypothesis on a "velocity set", a setback for applications. In this paper we show how the same result remains valid when the convexity assumption is replaced by a weaker assumption involving merely the mixed constraints. As in [6] we also consider pointwise set control constraints. Moreover, and again as in [6], a nonsmooth version of the positive linear independence of the gradients with respect to the control of the function defining the mixed constraints plays a key role in validation of our main result. The problem of interest is (P)

$$\left\{\begin{array}{lcll} \text{Minimize } l(x(0),x(1)) & & & \\ \text{subject to} & & & \\ \dot{x}(t) & = & f(t,x(t),u(t)) & \text{a.e. } t\in[0,1] \\ 0 & \geqslant & g(t,x(t),u(t)) & \text{a.e. } t\in[0,1] \\ u(t) & \in & U(t) & \text{a.e. } t\in[0,1] \\ (x(0),x(1)) & \in & C. & \end{array}\right.$$

Here the given functions $f\colon[0,1]\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n$, $g\colon[0,1]\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^m$, and multifunction $U\colon[0,1]\rightrightarrows\mathbb{R}^k$ describe the system dynamics and control constraints, while the given set $C\subset\mathbb{R}^n\times\mathbb{R}^n$ and function $l\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ specify the endpoint constraints and costs. Also, a *process* is a pair (x,u) comprising a function $x\in W^{1,1}([0,1];\mathbb{R}^n)$ and a measurable functions $u\colon[0,1]\to\mathbb{R}^k$. An *admissible process for* (P) is a process satisfying the constraints. Here $W^{1,1}(T;\mathbb{R}^n)$ denotes the space of absolutely continuous functions mapping T to $\mathbb{R}^n$. An admissible process $(\bar{x},\bar{u})$ is a *local minimizer* (also known as *weak minimizer*) for (P) if there exists $\delta'>0$ such that $l(\bar{x}(0),\bar{x}(1))\leqslant l(x(0),x(1))$ holds for all admissible processes (x,u) satisfying the following condition for almost every $t \in [0,1]$:

$$|x(t)-\bar{x}(t)| \leqslant \delta', \quad |u(t)-\bar{u}(t)| \leqslant \delta'. \qquad (1)$$

2 Preliminaries

For g in $\mathbb{R}^m$, inequalities like $g \leqslant 0$ are interpreted componentwise. We focus on a particular process $(\bar{x},\bar{w})$, and write $\bar{\phi}(t)$ instead of $\phi(t,\bar{x}(t),\bar{w}(t))$ for both $\phi=f$ and $\phi=g$.

Here and throughout, $\mathbb{B}$ represents the closed unit ball centered at the origin regardless of the dimension of the underlying space and $|\cdot|$ the Euclidean norm or the induced matrix norm on $\mathbb{R}^{p\times q}$. For each t in $[0,1]$ and some $\delta>0$, we define

$$T_\delta(t)=\bar{x}(t)+\delta\mathbb{B}=\{y\in\mathbb{R}^n \,:\, |y-\bar{x}(t)|\leqslant\delta\}\,. \qquad (2)$$

Likewise we set

$$U_\delta(t)=U(t)\cap(\bar{u}(t)+\delta\mathbb{B}). \qquad (3)$$

The *Euclidean distance function* with respect to a given set $A\subset\mathbb{R}^k$ is a function $d_A:\mathbb{R}^k\to\mathbb{R}$ defined as $d_A(y)=\inf\,\{|y-x| \,:\, x\in A\}$.

A function $h:[0,1]\to\mathbb{R}^p$ lies in $W^{1,1}([0,1];\mathbb{R}^p)$ if and only if it is absolutely continuous; in $L^1([0,1];\mathbb{R}^p)$ iff it is integrable; and in $L^\infty([0,1];\mathbb{R}^p)$ iff it is essentially bounded. The norm of $L^1([0,1];\mathbb{R}^p)$ is denoted by $\|\cdot\|_1$ and the norm of $L^\infty([0,1];\mathbb{R}^p)$ is $\|\cdot\|_\infty$.

We make use of standard concepts from nonsmooth analysis. Let $A\subset\mathbb{R}^k$ be a closed set with $\bar{x}\in A$. The *limiting normal cone to A at $\bar{x}$* is denoted by $N_A(\bar{x})$. Given a lower semicontinuous function $f:\mathbb{R}^k\to\mathbb{R}\cup\{+\infty\}$ and a point $\bar{x}\in\mathbb{R}^k$ where $f(\bar{x})<+\infty$, $\partial f(\bar{x})$ denotes the *limiting subdifferential* of f at $\bar{x}$. When the function f is Lipschitz continuous near x, the convex hull of the limiting subdifferential, co $\partial f(x)$, coincides with the *(Clarke) subdifferential*. Properties of Clarke's subdifferentials (upper semi-continuity, sum rules, etc.), can be found in [1]. For details on such nonsmooth analysis concepts, see [1], [16], [18] (in infinite dimensions see also [12]).

3 Auxiliary Results: Scalar Case

We now focus on a special case of problem (P) when the number of inequality mixed constraints is one, i.e., $g:[0,1]\times\mathbb{R}^n\times\mathbb{R}^{k_u}\times\mathbb{R}^{k_v}\to\mathbb{R}$. Define the function

$$g^+(t,x,u)=\max\{0,\, g(t,x,u)\}\,.$$

The following two sets of hypotheses on the data of this special case of problem (P), which make reference to a parameter $\delta>0$ and a reference process $(\bar{x},\bar{u})$, will be of importance:

(H1) For each $(x,u)\in\mathbb{R}^n\times\mathbb{R}^k$, the function $t\to(f(t,x,u),g(t,x,u))$ is Lebesgue measurable. Also, there exists a function $L\in L^1$ such that both $\phi=f$ and $\phi=g$ obey this inequality for almost every t in $[0,1]$:

$$|\phi(t,x,u)-\phi(t,x',u')|\leqslant L(t)\,|(x,u)-(x',u')|$$

for all $x,x'\in T_\delta(t),\ u,u'\in\mathbb{R}^k$.

(H2) The multifunction U has Borel measurable graph. For all $\delta>0$ sufficiently small, the set $U_\delta(t)$, as defined in (3), is closed for almost every $t\in[0,1]$.

(H3) The endpoint constraint set C is closed; the cost function l is locally Lipschitz in a neighbourhood of $(\bar{x}(0),\bar{x}(1))$.

(H4) Both $K_f(t) := |f(t,\bar{x}(t),\bar{u}(t))|$ and $K_g(t):=|g(t,\bar{x}(t),\bar{u}(t))|$ are integrable on $[0,1]$.

(H5) There exist a constant $K_1>0$ and a function $h\in L^\infty([0,1];\mathbb{R}^k)$, with $|h(t)|=1$ a.e., such that the following condition is satisfied for almost every $t\in[0,1]$, all $(x,u)\in T_\delta(t)\times U_\delta(t)$ for which $g(t,x,u)\geqslant 0$ and all vectors $(\gamma,\psi)\in$ co $\partial_{x,u}g(t,x,u)$:

$$\psi^j\cdot h(t)\geqslant K_1.$$

Consider additionally the following convexity assumption:

(CC) For almost every $t\in[0,1]$, each of the following sets is convex:
$$V^+(t,x)=\{(f(t,x,u),g^+(t,x,u)+s)\colon u\in U_\delta(t),\ s\geqslant 0\}\,.$$

In the context of the special case of problem (P) under consideration, the unmaximized Hamiltonian is the function $H: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ defined as $H(t,x,p,r,u) := p \cdot f(t,x,u) + r \cdot g(t,x,u)$.

Applying the weak nonsmooth Maximum Principle given by Theorem 4.1 in [6] to this special case of (P) we obtain the proposition stated below. Clearly the main setback to the application of this proposition is the restrictive nature of hypothesis CC.

Proposition 3.1 *Let $(\bar{x},\bar{u})$ be a local minimizer for problem (P), with $m=1$. Assume H1–H5 and CC. Then there exist an absolutely continuous function $p: [0,1] \to \mathbb{R}^n$, integrable functions $\xi: [0,1] \to \mathbb{R}^k$, and $r: [0,1] \to \mathbb{R}^m$, and a scalar $\lambda \geqslant 0$ such that*

$$\|p\|_\infty + \lambda > 0,$$

$$(-\dot{p}(t), \xi(t)) \in \operatorname{co} \partial_{x,u} H(t, \bar{x}(t), p(t), r(t), \bar{u}(t))$$

$$\xi(t) \in \beta(t) \operatorname{co} \partial d_{U_\delta}(\bar{u}(t)) \text{ a.e. } t,$$

$$r(t) \cdot g(t, \bar{x}(t), \bar{u}(t)) = 0 \textit{ and } r(t) \leqslant 0 \text{ a.e. } t,$$

$$(p(0), -p(1)) \in N_C(\bar{x}(0), \bar{x}(1)) + \lambda \partial l(\bar{x}(0), \bar{x}(1)),$$

where β depends only on δ, $L, K_f \in L^1([0,1];\mathbb{R})$ and the Lipschitz constant of l.

4 Scalar Case Without Convexity

Recall that we are considering (P) with scalar mixed constraints, that is, we assume that the number of inequality mixed constraints is one (i.e., $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^{k_u} \times \mathbb{R}^{k_v} \to \mathbb{R}$).

Consider the following condition:

(INT) For almost every t in $[0,1]$, we have $\{u \in U_\delta(t) : g(t,x,u) = 0\} \neq \emptyset$, for all $x \in T_\delta(t)$.

Proposition 4.1 *Let $(\bar{x},\bar{u})$ be a local minimizer for problem (P) when $m=1$ (i.e., $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$). Assume H1–H5 and INT. Then there exist an absolutely continuous function $p: [0,1] \to \mathbb{R}^n$, an integrable function $\xi: [0,1] \to \mathbb{R}^k$, an integrable function $r: [0,1] \to \mathbb{R}$, and a scalar $\lambda \geqslant 0$ such that (4)–(4) are satisfied.*

The proof of this proposition is presented in section 6.

5 Vector Valued Mixed Constraints

We now focus on the general problem (P) when the number of mixed constraints is greater than 1, i.e., we consider $g: [0,1] \times \mathbb{R}^n \mathbb{R}^k \to \mathbb{R}^m$, with $m > 1$.

Define the function $g^+(t,x,u) = \max\{g_1(t,x,u), \ldots, g_m(t,x,u)\}$. Before establishing our main result we consider two extra hypotheses:

(H5V) There exist a constant $K_1 > 0$ and a function $h \in L^\infty([0,1];\mathbb{R}^k)$, with $|h(t)| = 1$ a.e., such that the following condition is satisfied for almost every $t \in [0,1]$, all $(x,u) \in T_\delta(t) \times U_\delta(t)$, all $j \in \{1,\ldots,m\}$ for which $g_j(t,x,u) \geqslant 0$ and all vectors $(\gamma^j, \psi^j) \in \operatorname{co} \partial_{x,u} g_j(t,x,u)$:

$$\phi^j \cdot h(t) \geqslant K_1.$$

(H6V) For almost every t in $[0,1]$, we have $\{u \in U_\delta(t) : g^+(t,x,u) = 0\} \neq \emptyset$, for all $x \in T_\delta(t)$.

Observe that H5V is a vector valued version of the H5 and that H6V is an adaptation of the "interiority" hypothesis INT consider in the subsection . Hypothesis H6V states that for pairs (x,u) closed to the local minimizer $(\bar{x}(t), \bar{u}(t))$ there exists at least a component of the function g defining the mixed constraints that touches the boundary of the admissible set. On the other hand, hypothesis H5V is a nonsmooth version of regularity assumptions on the mixed constraints. Assuming smoothness of the function g, H5V coincides with the well known positive linear independence of the gradients $\nabla_v g_i$. In this respect we refer the reader to [8], [15], [4] and [5]).

Theorem 5.1 *Let $(\bar{x},\bar{u})$ be a local minimizer for problem (P), with $m \geqslant 1$. Assume H1–H4, H5V and H6V. Set $H(t,x,p,r,u) = p \cdot f(t,x,u) + r \cdot g(t,x,u)$. Then there exist an absolutely continuous function $p: [0,1] \to \mathbb{R}^n$, integrable functions $\xi: [0,1] \to \mathbb{R}^k$, and $r: [0,1] \to \mathbb{R}^m$, and a scalar $\lambda \geqslant 0$ such that*

$$\|p\|_\infty + \lambda > 0,$$

$$(-\dot{p}(t), \xi(t)) \in \operatorname{co} \partial_{x,u} H(t, \bar{x}(t), p(t), r(t), \bar{u}(t)) \text{ a.e. } t,$$

$$\xi(t) \in \operatorname{co} N_{U_\delta}(\bar{u}(t)) \text{ a.e. } t,$$

$$r(t) \cdot g(t, \bar{x}(t), \bar{u}(t)) = 0 \textit{ and } r(t) \leqslant 0 \text{ a.e. } t,$$

$$(p(0), -p(1)) \in N_C(\bar{x}(0), \bar{x}(1)) + \lambda \partial l(\bar{x}(0), \bar{x}(1)).$$

This Theorem results as a corollary of the Proposition 4.1. The details can be found in [7] and are omitted.

We would like to add that the above Theorem generalizes the main result in [6] to cover nonconvex problems. In contrast to Theorem 3.1 in [5] this Theorem now covers problems with possibly nonsmooth mixed inequality constraints and, remarkably, pointwise set control constraints. All the hypotheses under which Theorem 5.1 holds can be seen as direct adaptation of the hypotheses impose in [5] with the exception of H6V. Indeed, H6V is an hypothesis without direct analogous in the literature of necessary conditions for mixed constraints optimal control problems.

6 Proposition 4.1 Proof Outline

The proof breaks in three steps. We first prove the theorem under the interim hypotheses

(IH) For almost every $t \in [0,1]$, each of the following sets is convex:

$$F(t,x) = \{(f(t,x,u), g(t,x,u)) : u \in U_\delta(t)\}$$

for all $x \in T_\delta(t)$.

(ECS) $C = C_0 \times \mathbb{R}^n$ where $C_0 \subset \mathbb{R}^n$ is a closed set.

Step 1: *Show that INT and IH imply CC.* Take any $u, u' \in U_\delta(t)$, $s, s' \geqslant 0$ and $\alpha \in [0,1]$. Set $\widetilde{s} = \alpha s + (1-\alpha)s'$ and

$$A = \alpha g^+(t,x,u) + (1-\alpha) g^+(t,x,u').$$

We show that there exists $\widetilde{u} \in U_\delta(t)$ such that

$$A + \widetilde{s} = g^+(t,x,\widetilde{u}) + \widetilde{s}, \tag{4}$$

that is, CC holds. Consider three cases:

1. If $g(t,x,u), g(t,x,u') \geqslant 0$, then $A = \alpha g(t,x,u) + (1-\alpha) g(t,x,u')$ and by IH there exists a $\widetilde{w} \in W_\delta(t)$ such that (4) holds.

2. If $g(t,x,u), g(t,x,u') < 0$, then $g^+(t,x,u) = g(t,x,u') = 0$. By INT there exist $u_1, u_1' \in U_\delta(t)$ such that $g(t,x,u_1)$ and $g(t,x,u_1')$ are both zero. Thus $A = \alpha g(t,x,u_1) + (1-\alpha) g(t,x,u_1') = 0$ and, by IH, there exists a $\widetilde{u} \in U_\delta(t)$ such that (4) holds.

3. Suppose that $g(t,x,u) < 0$ and $g(t,x,u') \geqslant 0$. By INT there exists a $\widehat{u} \in U_\delta(t)$ such that $g(t,x,\widehat{u}) = 0$ and then $g^+(t,x,\widehat{u}) = 0$. Thus $A = \alpha g(t,x,\widehat{u}) + (1-\alpha) g(t,x,u')$. Then (4) follows from IH.

Next, an appeal to Proposition 3.1 permit us to deduce that Proposition 4.1 holds when INT and IH replace CC. Notice that the fact that g is a scalar valued function is essential.

Step 2: *Removal of IH.*

By adjusting $\delta > 0$ we can arrange that $(\bar{x}, \bar{u})$ is a local minimizer over all admissible processes (x,u) for (P) such that $\|x - \bar{x}\|_\infty \leqslant \delta$ and $\|u - \bar{u}\|_\infty \leqslant \delta$. Under our assumptions the set $F(t,x)$, as defined in IH, is nonempty for each (t,x) in the set

$$\Omega = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t \in [0,1],\, x \in \bar{x}(t) + \delta\mathbb{B}\}.$$

Write

$$R := \{x \in W^{1,1} : x(0) \in C_0, \} \\ (\dot{x}(t), 0) \in F(t, x(t))\}.$$

By the Generalized Filippov Selection Theorem [18, Thm.2.3.13]),$\bar{x}$ is a minimizer for the problem

$$\begin{cases} \text{Minimize } l(x(0), x(1)) \\ \text{over arcs } x \text{ in } R \text{ satisfying } \|x - \bar{x}\|_{L^\infty} < \delta. \end{cases}$$

A straightforward modification of the proof of the Relaxation Theorem (see, e.g., [18, Thm.2.7.2]) implies that any arc x in the set

$$R_r := \{x \in W^{1,1} : x(0) \in C_0, \\ (\dot{x}(t), 0) \in \operatorname{co} F(t, x(t))\}$$

which satisfies $\|x - \bar{x}\|_{L^\infty} < \delta$ can be approximated by an arc y in R satisfying $\|y - \bar{x}\|_{L^\infty} < \delta$. The continuity of the mapping

$$x \to l(x(0), x(1))$$

on a neighbourhood of $\bar{x}$ implies that $\bar{x}$ is a minimizer for the optimization problem

$$\begin{cases} \text{Minimize } l(x(0), x(1)) \\ \text{over arcs } x \in R_r \text{ satisfying } \|x - \bar{x}\|_{L^\infty} < \delta. \end{cases}$$

By the Generalized Filippov Selection Theorem and Carathéodory's Theorem,

$$\{\bar{x}, \bar{y} \equiv l(\bar{x}(0), \bar{x}(1)), (\bar{u}_0, \ldots, \bar{u}_n) \equiv \\ (\bar{u}, \ldots, \bar{u}), (\lambda_0, \lambda_1, \ldots, \lambda_n) \equiv (1, 0, \ldots, 0)\}$$

is a minimizer for the optimization problem (C)

$$\begin{cases} \text{Minimize } y(1) \\ \text{over } x \in W^{1,1},\ y \in W^{1,1}, \text{and measurable} \\ \text{functions } u_0,\ldots,u_n,\ \lambda_0,\ldots,\lambda_n \\ \text{ satisfying} \\ \dot{x}(t) \ = \ \sum_i \lambda_i(t) f(t,x(t),u_i(t)), \text{ a.e.,} \\ \quad \dot{y}(t) \ = \ 0, \text{ a.e.} \\ 0 \ \geqslant \ \sum_i \lambda_i(t) g(t,x(t),u_i(t)), \quad \text{a.e.} \\ (\lambda_0(t),\ldots,\lambda_n(t)) \in \Lambda, \\ u_i(t) \in U_\delta(t),\ i=0,\ldots,n \quad \text{a.e.} \\ (x(0),x(1),y(0)) \ \in \ \text{epi}\left\{\widetilde{l} + \Psi_{C_0\times\mathbb{R}^n\times\mathbb{R}}\right\}. \end{cases}$$

Here $\widetilde{l}(x,x',y) = l(x,x')$, Ψ_A is the indicator function of the set A, ($\Psi_A(z) = 0$ if $x \in D$ and $\Psi_A(z) = +\infty$ otherwise),

$$\Lambda \ := \ \{\lambda'_0,\ldots,\lambda'_n : \lambda'_i \geqslant 0 \text{ for } i=0,\ldots,n \\ \text{and } \sum_{i=0}^{n} \lambda'_i = 1\right\},$$

and $(\lambda_0,\ldots,\lambda_n)$, $(u_0,\ldots,u_n)$ are regarded as control variables. This is a problem **with a scalar mixed constraint** to which the proposition, as proved in the last step, applies. We can now write the optimality conditions for this problem work on them to get the result for the original problem without convexity. For the details we refer to [].

Step 3: *Validation of the result obtained in Step 2 when hypothesis ECS is removed.*

Let D denote the set of pairs (u,a,b) such that $u\colon[0,1]\to\mathbb{R}^k$ is a measurable function and $(a,b)\in C$ for which there exist absolutely continuous functions (x,y) such that

$$\begin{cases} \dot{x}(t) & = & f(t,x(t),u(t)), \quad \text{a.e.}\, t \\ \dot{y}(t) & = & 0, \quad \text{a.e.}\, t \\ 0 & \geqslant & g(t,x(t),u(t)), \quad \text{a.e.}\, t \\ u(t) & \in & U_\delta(t), \quad \text{a.e.}\, t \\ (x(t),y(t)) & \in & T_\delta(t)\times T_\delta(1) \quad \text{for all } t \\ (x(0),y(0)) & \in & C \end{cases}$$

We provide D with the metric

$$\Delta((u,u'),(a,a'),(b,b') = \\ \int_0^1 \left|u(t)-u'(t)\right| dt + \left|a-a'\right| + \left|b-b'\right|,$$

Choose a sequence ε_i such that $\varepsilon_i \downarrow 0$ and $\sum \varepsilon_i < +\infty$, and, for each i, define the function

$$l_i(x,y,x',y') = \\ \max\left\{l(x,y) - l(\bar{x}(0),\bar{x}(1)) + \varepsilon_i^2,\ \ |x'-y'|\right\}.$$

Consider

$$(R_i) \quad \begin{cases} \text{Minimize} & l_i(a,b,x(1),b) \\ \text{subject to} & (u,a,b)\in D. \end{cases}$$

Since $(\bar{u},\bar{x}(0),\bar{x}(1))\in D$, D is nonempty. It is a simple matter to check that (D,Δ) is a complete metric space on which the functional $l_i\colon D\to\mathbb{R}$ is continuous.

Notice that $l_i(\bar{x}(0),\bar{x}(1),\bar{x}(1),\bar{x}(1)) = \varepsilon_i^2$. Since $l_i \geqslant 0$, it follows that the process $(\bar{u},\bar{x}(0),\bar{x}(1))$ is a "ε_i^2-minimizer" for (R_i). Then we apply Ekeland's Variational Principle (Theorem 3.3.1 in [18]). Rewriting the conclusions in control theoretical terms we obtain a sequence of perturbed problems to which the necessary conditions obtained in the previous step hold. Taking limits we obtain the required conclusions.

Referências

[1] F. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley, New York, 1983.

[2] M. R. de Pinho and R. B. Vinter, An Euler-Lagrange inclusion for optimal control problems, *IEEE Trans. Automat. Control*, vol 40, 1995, pp 1191-1198.

[3] M. R. de Pinho and A. Ilchmann, Weak maximum principle for optimal control problems with mixed constraints, *Nonlinear Anal.*, vol 48, 2002, pp 1179-1196.

[4] F. Clarke, The maximum principle in optimal control, then and now, *Control Cybernet.*, vol. 34, 2005, pp 709-722.

[5] M.d.R. de Pinho and J. F. Rosenblueth, Necessary Conditions for Constrained Problems under Mangasarian-Fromowitz Conditions*SIAM J. Control Optim.*, vol 47, 2008, pp 535-552.

[6] M.d.R. de Pinho, P. Loewen and G. N. Silva, A weak maximum principle for optimal control problems with nonsmooth mixed constraints, *submitted to Set Valued Analysis*, 2008.

[7] M.d.R. de Pinho and G. N. Silva, Optimal Control Problems with Nonsmooth and Mixed Constrained Control Problems without Convexity Assumptions, *In preparation*, 2008.

[8] A. V. Dmitruk, Maximum principle for the general optimal control problem with phase and regular mixed constraints, *Comput. Math. Model.*, vol 4, 1993, pp 364-377.

[9] M. R. Hestenes, *Calculus of Variations and Optimal Control Theory*, John Wiley, New York, 1966.

[10] E.N. Devdaryani and Y. S. Ledyaev , Maximum principle for implicit control systems, *Appl. Math. Optim.*, vol. 40, 1996, pp 79-103.

[11] A. A. Milyutin and N. P. Osmolovskii, *Calculus of Variations and Optimal Control*, Translations of Mathematical Monographs 180, American Mathematical Society, Providence, Rhode Island, 1998.

[12] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation I. Applications*, Fundamental Principles of Mathematical Sciences 330, Springer-Verlag, Berlin, 2006.

[13] L. W. Neustadt, *Optimization, A Theory of Necessary Conditions*, Princeton University Press, New Jersey, 1976.

[14] Z. Páles and V. Zeidan, First- and second-order necessary conditions for control problems with constraints, *Trans. Amer. Math. Soc.*, vol. 346, 1994, pp 421–453.

[15] Z. Páles and V. Zeidan, Optimal control problems with set-valued control and state constraints, *SIAM J. Optim.*, vol 14, 2003, pp 334-358.

[16] R. T. Rockafellar and B. Wets, *Variational Analysis*, Grundlehren Math. Wiss. 317, Springer-Verlag, Berlin, 1998.

[17] G. Stefani and P. L. Zezza, Optimality conditions for a constrained control problem, *SIAM J. Control Optim.*, vol 34, 1996, pp 635-659.

[18] Richard Vinter, *Optimal Control*, Systems Control Found. Appl., Birkhäuser, Boston, 2000.