On the Dynamics of the Mixmaster Universe System

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Abstract: *In this work we study the flow on three invariant sets of dimension five for the classical mixmaster universe system. In these invariant sets, using the Darboux theory of integrability, we prove the non–existence of periodic solutions and we study their dynamics. Moreover, we find three invariant sets of dimension four where the flow is integrable.*

1 Introduction

The mixmaster universe model, also known, as the Bianchi IX model, is obtained through a convenient solution of Einstein's equations and corresponds to the Hamiltonian system

$$
\begin{array}{rcl}
\dot{q}_1 &=& 12q_1(p_1q_1 - p_2q_2 - p_3q_3),\\
\dot{q}_2 &=& 12q_2(-p_1q_1 + p_2q_2 - p_3q_3),\\
\dot{q}_3 &=& 12q_3(-p_1q_1 - p_2q_2 + p_3q_3),\\
\dot{p}_1 &=& -12p_1(p_1q_1 - p_2q_2 - p_3q_3)-\\
&& \frac{1}{3}(q_1 - q_2 - q_3),\\
\dot{p}_2 &=& -12p_2(-p_1q_1 + p_2q_2 - p_3q_3)-\\
&& \frac{1}{3}(-q_1 + q_2 - q_3),\\
\dot{p}_3 &=& -12p_3(-p_1q_1 - p_2q_2 + p_3q_3)-\\
&& \frac{1}{3}(-q_1 - q_2 + q_3),
\end{array} \tag{1}
$$

in $\mathbb{R}^6$ with three degrees of freedom and with zero energy, i.e. $G = 0$ for the Hamiltonian

$$
\begin{array}{rcl}
G &=& 6(p_1^2q_1^2 + p_2^2q_2^2 + p_3^2q_3^2 - 2p_1q_1p_2q_2\\
&& +2p_1q_1p_3q_3 - 2p_2q_2p_3q_3) + \frac{1}{6}(q_1^2+\\
&& q_2^2 + q_3^2 - 2q_1q_2 - 2q_1q_3 - 2q_2q_3).
\end{array} \tag{2}
$$

Of course $\dot{q}_i = G_{p_i}$ and $\dot{p}_i = -G_{q_i}$ for $i = 1, 2, 3$. The function G is a *first integral* of system (1); i.e, it is a function which is constant over the trajectories of this system. As usual the dots in system (1) denote derivative with respect to the time t.

This model has attracted the interest of both cosmologists and integrability specialists. See for instance [2, 4, 5, 6, 7, 9, 10, 11, 12].

Consider the coordinate change defined by

$$y_i = q_i, \quad z_i = p_iq_i,$$

for $i = 1, 2, 3$. In these coordinates system (1) becomes the following homogeneous polynomial differential system of degree 2 in $\mathbb{R}^6$:

$$
\begin{array}{rcl}
\dot{y}_1 &=& y_1(z_1 - z_2 - z_3),\\
\dot{y}_2 &=& y_2(-z_1 + z_2 - z_3),\\
\dot{y}_3 &=& y_3(-z_1 - z_2 + z_3),\\
\dot{z}_1 &=& -y_1(y_1 - y_2 - y_3),\\
\dot{z}_2 &=& -y_2(-y_1 + y_2 - y_3),\\
\dot{z}_3 &=& -y_3(-y_1 - y_2 + y_3),
\end{array} \tag{3}
$$

and the first integral G can be written now as

$$
\begin{array}{rl}
H = & (z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3-\\
& 2z_2z_3)/2 + (y_1^2 + y_2^2 + y_3^2 - 2y_1y_2-\\
& 2y_1y_3 - 2y_2y_3)/2.
\end{array} \tag{4}
$$

For $i = 1, 2, 3$ the hyperplane $y_i = 0$ is invariant by the flow of system (3) and z_i is a first integral on the hyperplane $y_i = 0$. Here an *invariant set* under the flow of system (3) means that if an orbit of this system has a point on this set, then the whole orbit is contained into the set.

We observe that the equations of system (3) are invariant by the permutation

$$(y_1, y_2, y_3, z_1, z_2, z_3) \mapsto (y_2, y_3, y_1, z_2, z_3, z_1) \mapsto (y_3, y_1, y_2, z_3, z_1, z_2). \tag{5}$$

Therefore, to know the dynamics on the hyperplane $y_1 = 0$ is equivalent to know it at any hyperplane $y_i = 0$ for $i = 1, 2, 3$. Hence in what follows we only study the dynamics on the hyperplane $y_1 = 0$, and only state explicitely the results for this hyperplane.

For every $c \in \mathbb{R}$ the solutions of system (3) restricted to the invariant 4–dimensional hyperplane of codimension 2

$$\Delta = \{(y_1 = 0, y_2, y_3, z_1 = c, z_2, z_3) \in \mathbb{R}^6\}$$

are given by the solutions of the system

$$\begin{array}{rcl} \dot{y}_2 & = & y_2(z_2 - z_3 - c), \\ \dot{y}_3 & = & y_3(-z_2 + z_3 - c), \\ \dot{y}_4 & = & -y_2(y_2 - y_3), \\ \dot{y}_5 & = & -y_3(-y_2 + y_3). \end{array} \tag{6}$$

Let $s < 4$. The functions $F_1, ..., F_s : \Delta \to \mathbb{R}$ are *independent* if the $s \times 4$ matrix

$$\frac{\partial(F_1, ..., F_s)}{\partial(y_2, y_3, z_2, z_3)}$$

has rank s at all points $(0, y_2, y_3, c, z_2, z_3) \in \Delta$, except perhaps on a subset of Δ of Lebesgue measure zero.

Our main results are the following three theorems.

Theorem 1. *If $c = 0$, then system (6) is integrable (i.e. it has three independent first integrals). Moreover we provide the explicit expression of its solutions.*

We observe here that if a system is integrable, then we can obtain its orbits simply performing the intersections of the level sets of its first integrals.

Let $\varphi(t) = \varphi(t, p)$ be the solution of system (6) passing through the point $p = (0, \overline{y_2}, \overline{y_3}, c, \overline{z_2}, \overline{z_3}) \in \Delta$, defined on its maximal interval $I_p = (\varpi_-(p), \varpi_+(p))$. If $\varpi_+(p) = \infty$ then the ϖ*–limit set* of p is

$$\varpi(p) = \{q \in \Delta : \exists \{t_n\} \text{ such that } t_n \to \infty \text{ and } \varphi(t_n) \to q \text{ as } n \to \infty\}.$$

In the same way, if $\varpi_-(p) = -\infty$ the α*–limit set* of p is

$$\alpha(p) = \{q \in \Delta : \exists \{t_n\} \text{ such that } t_n \to -\infty \text{ and } \varphi(t_n) \to q \text{ as } n \to \infty\}.$$

Theorem 2. *Assume $c \neq 0$. If $y_1 = 0$ and $z_1 = c$, then system (6) defined on Δ has two invariant hyperplanes $y_2 = 0$ and $y_3 = 0$, and the invariant function $F(y_2, y_3, z_2, z_3, t) = y_2 y_3 e^{2ct}$ (i.e. a first integral depending on the time). Moreover the following statements hold.*

(a) If $c > 0$, the α–limit of the orbits are in the hyperplanes $y_2 = 0$ or $y_3 = 0$.

(b) If $c < 0$, the ω–limit of the orbits are in the hyperplanes $y_2 = 0$ or $y_3 = 0$.

(c) The function $H(y_2, y_3, z_2, z_3) = (z_2 - z_3)^2 - 2c(z_2 + z_3) + (y_2 - y_3)^2$ is a first integral.

We must mention that under the assumptions of Theorem 2 the dynamics on the invariant subspace $y_2 = 0$ is given by the 2–dimensional system

$$\dot{y}_3 = y_3(k + z_3), \qquad \dot{z}_3 = -y_3^2,$$

where k is a convenient constant. The orbits of this system are the level curves of its first integral $K = (y_3^2 + z_3^2)/2 + kz_3$. The dynamics on the subspace $y_3 = 0$ is similar. In short, the α–limits or the ω–limits of the orbits of statements (a) and (b) of Theorem 2 are well known.

Since the hyperplane $y_1 = 0$ is the union of the hyperplanes $y_1 = 0$, $z_1 = c$ with $c \in \mathbb{R}$, Theorems 1 and 2 provide information on the dynamics over the whole invariant hyperplane $y_1 = 0$.

Our last result states the non–existence of periodic orbits in the hyperplane $y_1 = 0$.

Theorem 3. *System (3) has no periodic orbits in the invariant hyperplane $y_1 = 0$.*

The work is organized as follows. In section 2 we prove Theorems 1 and 2, and in section 3 we prove Theorem 3.

2 On the dynamics over the invariant hyperplane $y_1 = 0$

In this section we will prove Theorems 1 and 2.

Proof of Theorem 1: Now $c = 0$. We have that $\{(b, b, 0, 0); b \in \mathbb{R}\}$ is a straight line of singular points of system (6). The eigenvalues of the linear part of system (6) at a singular point

$(b, b, 0, 0)$ are $0, 0, 2bi, -2bi$. To write the linear part of system (6) at $(0, 0, 0, 0)$ into its real Jordan canonical form we use the following change of coordinates

$$\begin{aligned} y_2 &= x_2 - x_4, \\ y_3 &= x_2 + x_4, \\ z_2 &= x_1 - x_3, \\ z_3 &= x_1 + x_3. \end{aligned} \tag{7}$$

In these coordinates system (6) with $c = 0$ becomes

$$\begin{aligned} \dot{x}_1 &= -2x_4^2, \\ \dot{x}_2 &= 2x_3x_4, \\ \dot{x}_3 &= -2x_2x_4, \\ \dot{x}_4 &= 2x_2x_3. \end{aligned} \tag{8}$$

This system restricted to the variables (x_2, x_3, x_4) is invariant and it is close to the system of the Euler–Lagrange equations of the rigid body with fixed center of mass. Then its solution is

$$\begin{aligned} x_2(t) &= A\sqrt{-k}\ \mathrm{sn}(2A(t + t_0)|k), \\ x_3(t) &= A\sqrt{-k}\ \mathrm{cn}(2A(t + t_0)|k), \\ x_4(t) &= A\ \mathrm{dn}(2A(t + t_0)|k), \end{aligned}$$

and by direct integration we get that

$$x_1(t) = B - \frac{E\big(\mathrm{am}\,(2A(t + t_0)|k)\,, k\big)\, x_4(t)}{\sqrt{1 - (x_2(t)/A)^2}},$$

where the functions sn, cn, dn are the Jacobi elliptic functions, and E is the elliptic integral of second kind. Of course, A, k, t_0 and B are constants of integration.

It is easy to check that the three functions

$$\begin{aligned} &F_1(x_1, x_2, x_3, x_4) = x_3^2 + x_4^2 = A^2k, \\ &F_2(x_1, x_2, x_3, x_4) = x_2^2 - x_4^2 = -A^2, \\ &F_3(x_1, x_2, x_3, x_4) = x_1 - \\ &\frac{x_4E\Big(\mathrm{am}\big(\mathrm{dn}^{-1}(x_4/\sqrt{x_4^2 - x_2^2}|k)|k\big)\,|k\Big)}{\sqrt{1 - x_2^2/(x_4^2 - x_2^2)}}, \end{aligned}$$

are first integrals of system (8). In this last equality $k = (x_3^2 + x_4^2)/(x_4^2 - x_2^2)$. We consider the four 3×3 minors of the 3×4 matrix

$$\partial(F_1, F_2, F_3)/\partial(x_1, x_2, x_3, x_4).$$

Since the intersection where the four minors are zero has Lebesgue measure zero, it follows that F_1, F_2 and F_3 are independent. ∎

Proof of Theorem 2: Now $c \neq 0$. System (6), after the change of coordinates (7), becomes

$$\begin{aligned} \dot{x}_1 &= -2x_4^2, \\ \dot{x}_2 &= 2x_3x_4 - cx_2, \\ \dot{x}_3 &= -2x_2x_4, \\ \dot{x}_4 &= 2x_2x_3 - cx_4. \end{aligned} \tag{9}$$

We observe that $f_1 = x_2 - x_4 = 0$ and $f_2 = x_2 + x_4 = 0$ are invariant hyperplanes. The cofactor of $f_1 = 0$ is $k_1 = -(2x_3 + c)$ and the cofactor of $f_2 = 0$ is $k_2 = 2x_3 - c$. Using the Darboux theory of integrability, see [3, 8], we get that

$$F(x_1, x_2, x_3, x_4, s) = (x_2^2 - x_4^2)e^{2ct}$$

is an invariant function of system (9); i.e. dF/dt over the orbits of the system is zero.

If $c > 0$ and $t \to -\infty$ then $F \to 0$. It means that $f_1f_2 \to 0$ and the α–limit of the orbits of system (9) are approaching the hyperplanes $f_1 = 0$ or $f_2 = 0$. So statement (a) of Theorem 1 is proved. If $c < 0$ and $t \to \infty$ then $F \to 0$. Again it means that $f_1f_2 \to 0$ and the ω–limit of the orbits of system (9) are approaching the hyperplanes $f_1 = 0$ or $f_2 = 0$. Hence this proves the statement (b) of Theorem 1.

Moreover, if we restrict the first integral H given by (4) to system (6), then we get the first integral of statement (c). This completes the proof of Theorem 2. ∎

3 About the non existence of periodic orbits

The goal of this section is to prove Theorem 3 when $y_1 = 0$ and $z_1 = c =$ constant for all $c \in \mathbb{R}$. The next two propositions establish the result for the cases $c = 0$ and $c \neq 0$, respectively.

Proposition 4. *If $y_1 = 0$ and $z_1 = 0$, then system (3) with $c = 0$ has no periodic orbits.*

Proof: We consider the change of coordinates (7). System (3) restricted to $y_1 = 0$ and $z_1 = 0$ becomes system (6). This system with $c = 0$ after the change of variables (7) becomes system (8). Since $\dot{x_1} = -2x_4^2 \leq 0$, it follows that system (8) has no periodic orbits except if they are contained in $x_4 = 0$. If such a periodic orbit exists, then over it $\dot{x_2} = 0$ and $\dot{x_3} = 0$. So, on the periodic orbit x_2 and x_3 are constant. Hence, from the fact that $\dot{x_4} = 2x_2x_3$ =constant, such a periodic orbit cannot exists. ∎

Proposition 5. *If $y_1 = 0$ and $z_1 = c \neq 0$, then system* (3) *with $c \neq 0$ has no periodic orbits.*

Proof: System (3) restricted to $y_2 = 0$ and $y_3 = c \neq 0$ is

$$\begin{array}{rcl} \dot{y_3} & = & y_3(-z_2 + z_3 - c), \\ \dot{z_2} & = & 0, \\ \dot{z_3} & = & -y_3^2. \end{array} \quad (10)$$

In order to investigate the existence of periodic orbit of system (3) on the hyperplane $y_1 = 0$, it is sufficient to study the existence for the (10). Since $\dot{z_3} = -y_3^2 \leq 0$, the periodic orbits only can exist if $y_3 = 0$, but then it would be formed by singular points, a contradiction. ∎

Proof of Theorem 3: According with the statements (a) and (b) of Theorem 2, if system (3) restricted to $y_1 = 0$ has a periodic orbit it must be contained into the hyperplanes $y_2 = 0$ or $y_3 = 0$. Therefore using Propositions 4 and 5 we conclude the proof. ∎

4 Conclusions

We have proved that the mixmaster universe system written into the form (3) has no periodic orbits on the three 5–dimensional invariant hyperplanes $y_k = 0$ for $k = 1, 2, 3$, see Theorem 3.

For every $k = 1, 2, 3$ we restrict our attention to the 4–dimensional invariant hyperplanes $y_k = 0$ and $z_k = c$ with $c \in \mathbb{R}$. If $c = 0$ then the mixmaster universe system restrict to these three 4–dimensional invariant hyperplanes is integrable, in the sense that we can compute explicitly their solutions, and in the sense that we provide three independent first integrals, see Theorem 1 and its proof. Moreover for a given $c \neq 0$ we show that the solutions on these three 4–dimensional invariant hyperplanes either start or end in the 3–dimensional invariant subhyperplanes planes $y_i = 0$ or $y_j = 0$ with i and j different from k, see Theorem 2. Additionally the flow on these subhyperplanes is easy to study.

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