Breeding and predictability in chaotic Dynamics

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Abstract: One of the basic tenets of science is making predictions. If we know previous behavior, how can we predict future behavior? Predictability is an indication of the instability of the underlying flow computed from a numerical model, where small errors in the initial conditions (or imperfections in the model) can grow to large amplitudes in finite times. Bred vectors are the difference between two nonlinear model integrations, periodically rescaled to avoid nonlinear saturation of the instabilities of interest. Here, bred vectors are applied for the Lorenz system, and a model for three coupling waves for solar activities connected to the space weather process. The bred vector growth can be used to predict which will be the last orbit in each of the two regimes and how long will the next regime last. The purpose of this paper is to describe the breeding method that explores chaotic model predictability and applied it to two dynamic systems.

1. Introduction

A desirable characteristic of any physical model is an ability to make predictions; such predictions are the lifeblood of climate science, for example. The predictability of weather and climate forecasts is determined by the projection of uncertainties in both initial conditions and model formulation onto flow-dependent instabilities of the chaotic climate attractor. Since it is essential to be able to estimate the impact of such uncertainties on forecast accuracy, no weather or climate prediction can be considered complete without a forecast of the associated flow-dependent predictability. Earth's climate is a prototypical chaotic system implying that its evolution is sensitive to the specification of the initial state. The prediction of the future state of a system knowing its initial conditions is a fundamental problem with obvious applications in geophysics flows. There are many limitations to the ability of predicting the state of a geophysical system. For example, how predictable is Earth's magnetosphere? With all emphasis on "space weather", this has become an important practical question. The question is also fundamental to understanding the dynamics of the magnetosphere [1].

Chaotic dynamics implies not only that forecast in sensitive to initial error, but also that rate of growth of initial error is itself a function of the initial state [11].

Bred vectors are the difference between two nonlinear model integrations, periodically rescaled. Toth and Kalnay [12] conjectured that they represent the same instabilities that generate the "errors of the day" and can therefore be used to estimate the shape of the forecast errors. The use of ensemble forecasting and data assimilation shows the importance of local predictability properties of the atmosphere in space and in time.

The method of applying small perturbations in chaotic systems has been applied to a variety of physical applications for some purposes. The "breeding method" to generate and properties of bred vectors in Section 2. In the next section, the breeding growth rates provide reliable forecast rules for regime transition in the Lorenz system, and in the simple model from solar physics dynamics: the three-wave model (a non-linear coupling three waves).

2. Breeding method

The "breeding method" is a well-established and computationally inexpensive method for generating perturbations for ensemble integrations. In examination of the local structure of the vectors indicates that there may be substantial redundancy when multiple independent breeding cycles are performed in parallel, and the vectors can be inefficient in spanning the range of locally growing perturbations. Breeding was developed as a method to generate initial perturbations for ensemble forecasting in numerical weather prediction at the National Centers for Environmental Prediction (NCEP) [13]. The method involves running the nonlinear model used for the control a second time, periodically subtracting the control from the perturbed solution, and rescaling the difference so that it has the same size as the original perturbation. The rescaled difference (a bred vector) is added to the control run and the process repeated. Their growth rate is a measure of the local instability of the flow.

Bred Vectors (BVs) are, by construction, closely related to Lyapunov vectors (LVs). BVs are different from LVs in two important ways: (a) bred vectors are never globally orthogonalized and are intrinsically local in space and time, and (b) they are finite amplitude, finite time vectors. Both assume an evolving basic solution $f(x,t)$ that satisfies the equation of a nonlinear model with a given discretization in space and time integration scheme: $f(x,t+\Delta t) = M(f(x,t))$. If the initial condition is perturbed, the linear evolution of perturbation is given by $\delta f(x,t+\Delta t) = L(f,t,\Delta t)\delta f(x,t)$ where the matrix $\mathrm{L} = \partial \mathrm{M} / \partial \mathrm{f}$ is the tangent linear model or propagator.

Bred Vectors (BVs) are computed as follows [7]:

1) Start with an arbitrary initial perturbation $\delta f(x,t)$ of size $\mathbf{A}$ defined with an arbitrary norm. This initialization step is executed only once. The size of $\mathbf{A}$ is essentially the only tunable parameter of breeding.
2) Add the perturbation to the basic solution, integrate the perturbed initial condition with the nonlinear model, and subtract the original unperturbed solution from the perturbed nonlinear integration:

$$\overline{\delta f(x,t+\Delta t)} = M[f(x,t)+\delta f(x,t)] - M[f(x,t)] \qquad (1)$$

3) Measure the size $A+\delta A$ of the evolved perturbation $\overline{\delta f(x,t+\Delta t)}$, and divide the perturbation by the measured amplification factor so that its size remains equal to A:

$$\delta f(x,t+\Delta t) = \overline{\delta f(x,t+\Delta t)} A / (A+\delta A) \qquad (2)$$

Steps (2) and (3) are repeated for the next time interval and so on. It has been found that after a short transient time of the order of the time scale of the dominant instabilities. In practical applications, bred vectors are intrinsically local in space and time, and they are finite amplitude, finite time vectors (Figure 1).

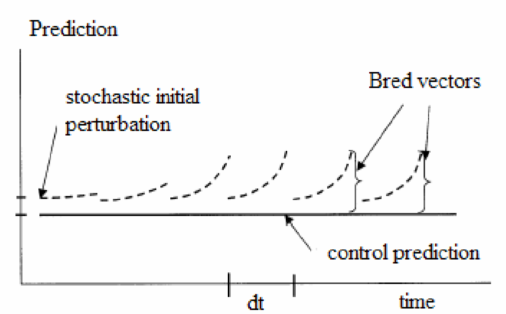

Figure 1-Schematic of the method to generate bred vectors (Evans et al, 2004).

3. **Numerical Experiments**

a) Lorenz System

Like an example to study bred vectors, we reproduce the Research Internships in Science and Engineering (RISE) experiment with the 3-variable Lorenz model that indicate that orthogonalizing the bred vectors can result in significantly improved performance. This experiment showed that the regime changes in Lorenz's model are predictable. The idea here is to explore the predictability of the Lorenz system [8] employing the breeding scheme. Lorenz system equations:

$$\frac{dx}{dt} = \sigma(y-x) \qquad (3a)$$

$$\frac{dy}{dt} = rx - y - xz \qquad (3b)$$

$$\frac{dz}{dt} = xy - bz \qquad (3c)$$

where the parameters $\sigma = 10$, b=8/3 and r=28 produce chaotic solutions (Figure 2). This model has been very widely used as a prototype of chaotic behavior. The model was integrated with a 4th order Runge-Kutta numerical scheme.

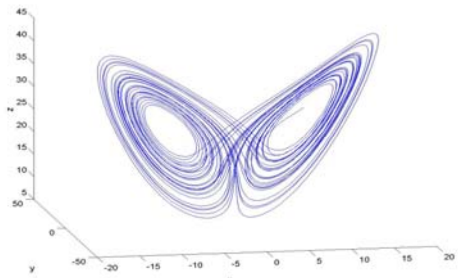

Figure 2- Solutions of the Lorenz model equations showing two chaotic regimes

Two Lorenz systems start with different initial condition. Firstly the Lorenz system is integrated with time steps Δt=0.01, and secondly started from an initial perturbation δx_0 added to the control at time t_0. Every 8 times steps the difference δx between the perturbed and the control run is took; the initial amplitude is rescaled and add it to the control. The growth rate of the perturbation was measured per time step as $g = \frac{1}{n} * \left(|\delta x| / |\delta x_0| \right)$ [5]. Figure 3 shows that procedure allows to estimate the stability of the attractor. Figure 4 shows the use of this simple procedure it allows us to estimate the stability of the attractor.

Examining the bred-vector growth for predictability (display the plot: growth rates versus *x*(*t*) – Figure 3 shows the growth rates (GR) with different color-star: red: GR>0.064, yellow: 0.032<GR<0.064, green: 0.00<GR<0.032, blue: GR<0.00) was possible to found simples rules to forecast [5]:

Rule-1: When the growth rate exceeds 0.064 over a period of eight steps, as indicated by the presence of one (or more) red stars, the current regime will end after it completes the current orbit.

Rule-2: The length of the new regime is proportional to the number of red stars (see Figure 4).

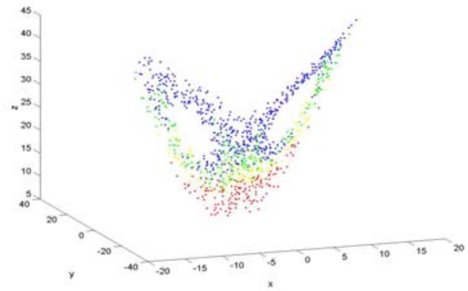

Figure 3 – a: The Lorenz classic attractor colored with the bred vector growth

The presence of a red star shows bred vector growth in the previous 8 steps was greater than 0.064. This rule can be used to forecast that the next orbit will be the last one in the current regime. The blue stars indicate a negative growth rate, meaning that the perturbations are actually decaying. The results shown in Figure 3 suggested that the bred vector growth would allow estimating of high and low predictability. The growth rates on the evolution of the variable *x*(*t*) provides a means to predict when the model will enter a new regime, and also how long the new regime will last.

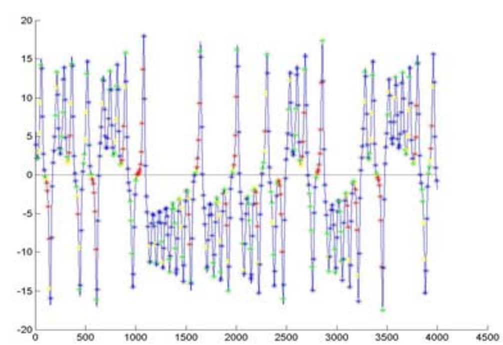

Figure 4 - *x*(*t*) for the classic Lorenz model with red stars providing "forecasting rules".

b) <u>Nonlinear Three-wave coupled model</u>

Nonlinear three-wave coupling is of general interest in many branches of physics such as nuclear fusion, space geophysics, astrophysics, nonlinear optics, and fluid mechanics. For example, it causes the stimulated scattering and anomalous absorption of laser beams in inertial fusion experiments [4]. This model appears in the plasma edge region of a magnetic fusion device during radio-frequency heating experiments, responsible for the generation and modulation of plasma waves in the planetary magnetosphere and solar wind [3].

The simplest model for describing the temporal dynamics of resonant nonlinear coupling of three waves (Langmuir, whistler, and kinetic Alfven waves) can be obtained assuming that the nonlinearity is sufficiently weak so that only quadratic terms in the wave amplitudes need to be considered [9]. In order for three-wave interactions to occur, the wave frequencies ω_α and wave vectors $\mathbf{k}_\alpha$ must satisfy the resonant conditions

$$\omega_3 \approx \omega_1 - \omega_2 \ , \ \ k_3 \approx k_1 - k_2 \qquad (4)$$

Under these circumstances, the nonlinear temporal dynamics of the system can be governed by the following set of three first-order autonomous differential equations written in terms of the complex slowly varying wave amplitude [10]:

$$\dot{A}_1 = v'_1 A_1 + A_2 A_3 \qquad (5a)$$

$$\dot{A}_2 = i\delta A2 + v'_2 A_2 - A_1 A_3^* \qquad (5b)$$

$$\dot{A}_3 = v'_3 A_3 - A_1 A_2^* \qquad (5c)$$

where the over-dot denotes differentiation with respect to the time like variable $\tau = \chi t$, χ is a characteristic frequency: $\delta=(\omega_1 - \omega_2 - \omega_3)/\chi$ is the normalized linear frequency mismatch and $v'_\alpha = v_\alpha/\chi$ give the linear wave behaviors on the long time scale. We assume here that the wave A_1 is linearly unstable ($v_1>0$) and the other two waves, A_2 and A_3, are linearly damped ($v'2=v'_3\equiv -v<0$) and henceforth we set $\chi=v_1$ so than $v'_1=1$ [10].

In the presence of linear wave growth and damping, the numerical solutions of the coupled three-wave system admit both periodic and chaotic waves. Figure 5 shows the attractor: one regime is in straight line, and other regime is the curve line.

The breeding method was applied on three-wave model like the first experiment: Firstly perform breeding on the three-wave model integrating with time steps $\Delta t=0.001$, and the second run started from an initial perturbation δx_0 added to the control at time t_0. Every 8 times steps (similar to the Lorenz system), the differences δx between the perturbed and the control run was collected and rescale them to the initial amplitude and add these to the control. The growth rate and the trajectory of wave A_1 were plotted. The rules were supposed like in the first experiment: **two or more red stars** indicates the regime transition, **two yellow stars** predict a short time for changing of another regime. The same rules are applied at waves A_2 and A_3. See Figures 6 –8.

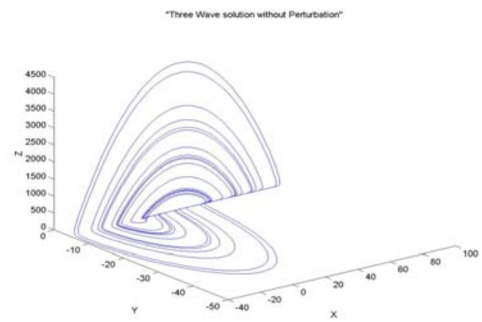

Figure 5- Solutions of the Three-wave model equations showing two chaotic regimes

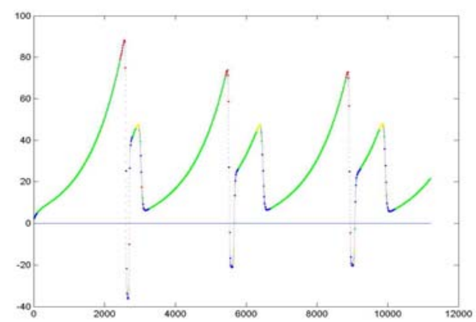

Figure 6 – $A_1(t)$ for the three-wave model with "forecasting rules".

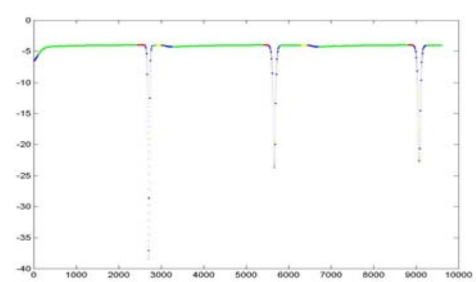

Figure 7 – $A_2(t)$ for the three-wave model with "forecasting rules".

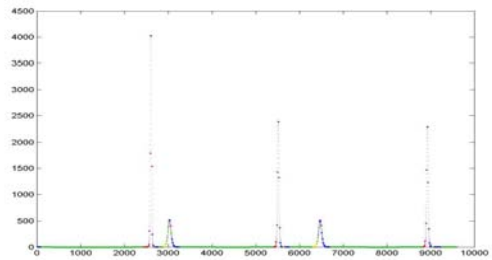

Figure 8 – $A_3(t)$ for the three-wave model with "forecasting rules".

3. **Conclusion**

We presented examples with the Lorenz models that indicate that with simple breeding, we can make accurate "long-range forecasts" of regime changes.

The chaotic behaviors of nonlinear interactions in three-wave model can be predicted by the analysis of bred vectors. This is an effective method of predict changes in nonlinear three-coupling waves in solar plasma physics.

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REFERENCES

[1] D.N. Baker, R.L. McPherron, T.E. Cayton, R.W. Klebsadel, Linear Prediction Filter analysis of relativistic electron properties at 6.6 R_E, J. Gephys. Res., vol. 95, pp. 15133-15140 (1990).

[2] A. C. L., Chian, F. A. Borotto, S. R. Lopes, J. R. Abalde, Chaotic Dynamics of Nonthermal Planetary Radio Emissions. Planetary and Space Science, vol. 48, pp. 9-21 (1999).

[3] A. C. L. Chian, F.B. Rizzato, Coupling of Electromagnetic Filamentation Instability and Electrostatic Langmuir Parametric-instabilities in Laser-Plasma Insteractions, J. Plasma Phys., vol. 51, pp. 61-73 (1994).

[4] C.C. Chow, A. Bers, A.K. Ram, Spatio-temporal Chaos in the Nonlinear 3-Wave Interaction, Phys. Rev. Lett., vol. 68, pp. 3379-3382 (1992).

[5] E. Evans, N. Bhatti, J. Kinney, L. Oann, M. Peña, S. Yang, E. Kalnay, RISE undergraduates find that regime changes in Lorens´s model are predictable. Bull. Amer. Meteo. Soc., Vol. 85, pp. 520-524 (2004).

[6] E. Kalnay, "Atmospheric Modeling, Data Assimilation and Predictability", Cambridge University Press, 2003.

[7] E. Kalnay, M. Peña, S.-C. Yang, M. Cai, Breeding and predictability in coupled Lorenz models, em "Proceedings of the ECMWF Seminar on Predictability", 9-13 September 2002.

[8] E. N. Lorenz, Deterministic non-periodic flow, *J. Atmos. Sci*., vol. 20, 130-141 (1963).

[9] S.R. Lopes, A.C-L. Chian, Controlling chaos in nonlinear three-wave coupling. *Phy. Rev*., vol. E 54, pp. 170-174 (1996).

[10] C. Meunier, M. N. Bussac, G. Laval, Intermittency at the onset of stochasticity in nonlinear resonant coupling process, Phys. D4, vol. 4, 236-243.

[11] T.N. Palmer, Extended-Range Atmospheric Prediction and the Lorenz Model, Bull. Amer. Meteo. Soc., Vol. 74, pp. 49–66 (1993).

[12] Z. Toth, E. Kalnay, Ensemble forecasting at NMC: The generation of perturbations, Bull. Amer. Meteo. Soc., vol. 74, pp. 2317-2330 (1993).

[13] Z. Toth, E. Kalnay, Ensemble forecasting at NCEP and the breeding method, *Mon. Wea. Rev.*, vol. 127, pp. 3297-3319 (1997).