Doubly-symmetric periodic solutions for planar analytic perturbations of Kepler problem: Application in the circle problem

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Abstract
Our main concern, in this work, is show analytically the existence of planar families doubly-symmetric periodic solutions close to circular or elliptic Keplerian orbits of analytic system which are symmetric perturbations of the Kepler problem. More precisely, our purpose is to study separately the existence of periodic solutions of the Hamiltonian system in two situations, namely: the first is associated the Hamiltonian function in the form

$$H = \frac{1}{2}\|\mathbf{p}\|^2 - \frac{1}{\|\mathbf{q}\|} - \epsilon^{\alpha} W_1(\mathbf{q}, \epsilon),$$

with $\mathbf{q} = (x, y) \in \mathbb{R}^2 \setminus \mathcal{K}$ where $\mathcal{K}$ is a compact set in $\mathbb{R}^2$ that contains the origin, $\mathbf{p} = (p_x, p_y) \in \mathbb{R}^2$, $\alpha > 0$ and W_1 is an analytic function. So, we are in the presence of a mechanical system. And, the second case the Hamiltonian function has the form

$$K = \tfrac{1}{2}(p_x^2 + p_y^2) - (xp_y - yp_x) - \frac{1}{\sqrt{x^2+y^2}} - \epsilon^{\alpha} W_2(x, y, \epsilon),$$

with $(x, y) \in \mathbb{R}^2 \setminus \mathcal{K}$ where again $\mathcal{K}$ is as before, $(p_x, p_y) \in \mathbb{R}^2$ and W_2 is an analytic function.

Such problems appear in the study of the several problems int the gravitational attraction in celestial mechanics (see [2]). In order to obtain our results, we prolong periodic orbits of the Kepler problem by Poincaré Continuation Method. But, the application of the Poincaré Continuation Method only is possible if we use the convenient variables. However, there are two sets of coordinates which make possible in a very simple way to describe symmetric periodic solutions: The Delaunay and Poincaré-Delaunay variables (see [5]).

In the literature, we can find some works that study the periodic orbits to homogeneous circle problem (see [1], [3] and [4]). We prove the existence of new families of doubly-symmetric periodic solutions as continuation of planar circular and elliptic Keplerian orbits with applications in the homogeneous circle problem.

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1 Introduction

In this work we show analytically the existence of planar families doubly-symmetric periodic solutions close to circular or elliptic Keplerian orbits of analytic system which are symmetric perturbations of the Kepler problem. More precisely, our purpose is to study separately the existence of periodic solutions of the Hamiltonian system in two situations, namely: the first is associated the Hamiltonian function in the form

$$H = \frac{1}{2}\|\mathbf{p}\|^2 - \frac{1}{\|\mathbf{q}\|} + \epsilon^{\alpha} W_1(\mathbf{q}, \epsilon), \qquad (1)$$

with $\mathbf{q} = (x, y) \in \mathbb{R}^2 \setminus \mathcal{K}$ where $\mathcal{K}$ is a compact set in $\mathbb{R}^2$ that contains the origin, $\mathbf{p} = (p_x, p_y) \in \mathbb{R}^2$, $\alpha > 0$ and W_1 is an analytic function. So, we are in the presence of a mechanical system. And, the second case the Hamiltonian function has the form

$$K = \tfrac{1}{2}(p_x^2 + p_y^2) - (xp_y - yp_x) - \frac{1}{\sqrt{x^2+y^2}} + \epsilon^{\alpha} W_2(x, y, \epsilon), \qquad (2)$$

with $(x, y) \in \mathbb{R}^2 \setminus \mathcal{K}$ where again $\mathcal{K}$ is as before, $(p_x, p_y) \in \mathbb{R}^2$ and W_2 is an analytic function. Notice that this system is not mechanical. In order to obtain the results, we use the direct application of the Poincaré Continuation Method.

By this reason one alternative in order to prolong periodic orbits of the Kepler problem, is the utilization of "goods coordinates" and to impose the doubly-symmetric condition in the system of the problem, with the objective to avoid degeneracy. However, there are two sets of coordinates which makes possible in a very simple way to describe symmetric periodic solutions. These coordinates are the Delaunay elements, which are well defined for elliptic orbits, and the Poincaré-Delaunay variables, which are useful (mainly) for circular orbits (see for example [5]).

2 Formulation of the problem

Initially we will consider the Hamiltonian function (1) where we will assume that the function W_1 is an analytic function in $\mathbf{q} \in \mathbb{R}^2 \setminus \mathcal{K}$ and ϵ. Since W_1 is analytic in ϵ, we can write W_1 in Taylor series around $\epsilon = 0$ and the Hamiltonian function (1) can be rewritten as

$$\begin{aligned} H(\mathbf{q}, \mathbf{p}, \epsilon) &= H_0(\mathbf{q}, \mathbf{p}) - \epsilon^{\alpha+i} H_1(\mathbf{q}) + \\ & \mathcal{O}(\epsilon^{\alpha+i+1}), \end{aligned} \tag{3}$$

where $H_0 = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{\sqrt{x^2+y^2}}$, and $H_1(\mathbf{q}) = \frac{1}{i!}\frac{\partial^i W_1(\mathbf{q},0)}{\partial \epsilon^i} \neq 0$, being i the first natural number with this property. Notice that the problem (3) consists in an analytic perturbation of the Kepler problem, and such problem is completely integrable and are very known its periodic solutions.

As second problem, we will consider the Hamiltonian system associated the Hamiltonian function (2). Writing W_2 in Taylor series in ϵ around $\epsilon = 0$, thus the Hamiltonian function (2) can be written as

$$K(\mathbf{q}, \mathbf{p}, \epsilon) = K_0(\mathbf{q}, \mathbf{p}) - \epsilon^{\alpha+i} K_1(q) + \mathcal{O}(\epsilon^{\alpha+i+1}), \tag{4}$$

where $K_0 = \frac{1}{2}(p_x^2 + p_y^2) - (xp_y - yp_x) - \frac{1}{\sqrt{x^2+y^2}}$ and $K_1(\mathbf{q}) = \frac{1}{i}\frac{\partial^i W_2(\mathbf{q},0)}{\partial \epsilon^i} \neq 0$, being i the first natural number with this property.

Our main aim will be continue circular and elliptic orbits of the Kepler problem for each of the previous situations. For the continuation of such solutions, we propose, the use of convenient variables according the case in study, because with this choose we will avoid problems of degeneracy.

2.1 The problem in Delaunay and Poincaré-Delaunay variables

Firstly, we will consider the Delaunay variables, which are defined by (see details in [5])

$$Q_1 = l, \quad P_1 = L, \quad Q_2 = g, \quad P_2 = G. \tag{5}$$

The Delaunay elements (l, g, L, G) are defined in an elliptic domain of the Kepler problem. As is habitual l is the mean anomaly, g the argument of the perigee measured from the x-axis, $L = \mathsf{a}^{1/2}$ is the semi-major axis of the ellipse, and $G = [\mathsf{a}(1 - e^2)]^{1/2}$ is the angular momentum. The variables l, g are angular variables modulus 2π, and L, G are radial variables.

It is verified that the Hamiltonian function of the Kepler problem in these coordinates is given by

$$H_0(\mathbf{Q}, \mathbf{P}) = -\frac{1}{2}\frac{1}{P_1^2}. \tag{6}$$

Thus, the Hamiltonian function (3) in the variables (5) becomes

$$\begin{aligned} H(\mathbf{Q}, \mathbf{P}, \epsilon) &= -\tfrac{1}{2P_1^2} - \epsilon^{\alpha+i} H_1(\mathbf{Q}, \mathbf{P}) + \\ & \mathcal{O}(\epsilon^{\alpha+i+1}), \end{aligned} \tag{7}$$

where

$$H_1(\mathbf{Q}, \mathbf{P}) = \frac{1}{i}\frac{\partial^i W_1(\mathbf{Q}, \mathbf{P}, 0)}{\partial \epsilon^i}, \tag{8}$$

where being i the first natural number such that $\frac{\partial^i W_1(\mathbf{Q},\mathbf{P},0)}{\partial \epsilon^i} \neq 0$. Then, follows that the equations of motion corresponding the Hamiltonian function (7) in the Delaunay coordinates (5) are

$$\begin{aligned} \dot{Q}_1 &= \frac{1}{P_1^3} & +\mathcal{O}(\epsilon^{\alpha+i}), \quad \dot{P}_1 = & \ 0 + \mathcal{O}(\epsilon^{\alpha+i}) \\ \dot{Q}_2 &= 0 & +\mathcal{O}(\epsilon^{\alpha+i}), \quad \dot{P}_2 = & \ 0 + \mathcal{O}(\epsilon^{\alpha+i}). \end{aligned} \tag{9}$$

In Delaunay coordinates (5) the Hamiltonian function (4) is given by

$$\begin{aligned} K(\mathbf{Q}, \mathbf{P}, \epsilon) &= -\tfrac{1}{2P_1^2} - P_2 + \epsilon^{\alpha+i} K_1(\mathbf{Q}, \mathbf{P}) + \\ & \mathcal{O}(\epsilon^{\alpha+i+1}), \end{aligned} \tag{10}$$

where $K_1(\mathbf{Q}, \mathbf{P}) = -\frac{1}{i}\frac{\partial^i W_2(\mathbf{Q},\mathbf{P},0)}{\partial \epsilon^i}$. The equations of motion of the problem (10) in Delaunay coordinates (5) assume the form

$$\begin{aligned} \dot{Q}_1 &= \frac{1}{P_1^3} & +\mathcal{O}(\epsilon^{\alpha+i}), \quad \dot{P}_1 = & \ 0 + \mathcal{O}(\epsilon^{\alpha+i}), \\ \dot{Q}_2 &= -1 & +\mathcal{O}(\epsilon^{\alpha+i}), \quad \dot{P}_2 = & \ 0 + \mathcal{O}(\epsilon^{\alpha+i}). \end{aligned} \tag{11}$$

Since Delaunay variables are not valid in a neighborhood of the circular orbits of the Kepler problem, in order to continue elliptic Keplerian orbits, we will use in this situation the Poincaré-Delaunay variables or sometimes called simply as Poincaré elements. In our approach we will use the following Poincaré-Delaunay variables

$$\begin{array}{llll} Q_1 = & l+g, & Q_2 = & -\sqrt{2(L-G)}\sin(g), \\ P_1 = & L, & P_2 = & \sqrt{2(L-G)}\cos(g), \end{array} \tag{12}$$

with l, g, L and G the orbital elements defined as before.

In this case the Hamiltonian function (3) (original variables) in the variables of Poincaré-Delaunay is given by

$$H(\mathbf{Q},\mathbf{P},\epsilon) = -\frac{1}{2P_1^2}+\epsilon^{\alpha+i}H_1(\mathbf{Q},\mathbf{P})+\mathcal{O}(\epsilon^{\alpha+i+1}), \tag{13}$$

where H_1 is as in (8) and the corresponding equations of motion corresponding in the Poincaré-Delaunay variables are

$$\begin{array}{llll} \dot{Q}_1 & = \frac{1}{P_1^3} & +\mathcal{O}(\epsilon^{\alpha+i}), & \dot{P}_1 = \quad 0+\mathcal{O}(\epsilon^{\alpha+i}) \\ \dot{Q}_2 & = 0 & +\mathcal{O}(\epsilon^{\alpha+i}), & \dot{P}_2 = \quad 0+\mathcal{O}(\epsilon^{\alpha+i}). \end{array} \tag{14}$$

While, in the Poincaré-Delaunay coordinates (12) the Hamiltonian function (4) has the form

$$\begin{array}{ll} K(\mathbf{Q},\mathbf{P},\epsilon) & = -\frac{1}{2P_1^2}-P_1+\frac{Q_2^2+P_2^2}{2}+ \\ & \epsilon^{\alpha+i}K_1(\mathbf{Q},\mathbf{P})+\mathcal{O}(\epsilon^{\alpha+i+1}), \end{array} \tag{15}$$

and the equations of associated Hamiltonian system, in the Poincaré-Delaunay coordinates (12) are

$$\begin{array}{lll} \dot{Q}_1 = \frac{1}{P_1^3} & +\mathcal{O}(\epsilon^{\alpha+i}), & \dot{P}_1 = 0+\mathcal{O}(\epsilon^{\alpha+i}), \\ \dot{Q}_2 = P_2 & +\mathcal{O}(\epsilon^{\alpha+i}), & \dot{P}_2 = -Q_2+\mathcal{O}(\epsilon^{\alpha+i}). \end{array} \tag{16}$$

Remark 2.1 *At the first the system of equations (9) and (14) like identical, because we have only written explicitly the unperturbed terms, and is clear that the Kepler problem in both coordinates are equal. In fact, the main difference between these two system are in the perturbed terms, i.e., in the terms $\mathcal{O}(\epsilon^{\alpha+i})$.*

Remark 2.2 *The Poincaré-Delaunay coordinates are well defined in a neighborhood of a circular orbit, and a such solution ($e=0$) occur when $L=G$. This last condition in the Poincaré-Delaunay variables can be expressed by the condition $Q_2=P_2=0$.*

2.2 Symmetries

In order to give necessary conditions to obtain symmetric periodic solutions and to characterize they in Delaunay and Poincaré- Delaunay variables, we will assume for a moment that the Hamiltonian function possesses the symmetries of reflection with respect to the x-axis and with respect to the y-axis, which we will denote by S_1 and S_2 respectively, i.e.,

$$\begin{array}{lll} S_1:(x,y,p_x,p_y) & \longrightarrow & (x,-y,-p_x,p_y), \\ S_2:(x,y,p_x,p_y) & \longrightarrow & (-x,y,p_x,-p_y). \end{array} \tag{17}$$

Clearly these symmetries fixed the x-axis and the y-axis, respectively. On the other hand, we know that by the Uniqueness of the solutions of the Ordinary Differential Equations, if one solution crosses orthogonally the respective axis of symmetry at $t=t_0$ and also in the time $t=T/4$, then such solution will be T-periodic. This kind of symmetric periodic solutions will be called doubly-symmetric, i.e., are symmetric in relation to both axis.

Next, we will characterize an orthogonal crossing in relation to the x-axis and in relation to the y-axis in terms of the Delaunay and Poincaré-Delaunay variables.

Lemma 2.1 *In the Delaunay variables (5), an orthogonal crossing in relation to the x-axis in $t=t_0$ is given by*

$$Q_1(t_0)=0\,(mod\,\pi), \quad Q_2(t_0)=0\,(mod\,\pi),$$

and in relation to the y-axis is given by

$$Q_1(t_0)=0\,(mod\,\pi), \quad Q_2(t_0)=\frac{\pi}{2}\,(mod\,\pi).$$

Lemma 2.2 *In the Poincaré-Delaunay (12) variables, an orthogonal crossing in relation to the x-axis in $t=t_0$ is given by*

$$Q_1(t_0)=0\,(mod\,\pi), \quad Q_2(t_0)=0,$$

and in relation to the y-axis is given by

$$Q_1(t_0)=\frac{\pi}{2}\,(mod\,\pi), \quad P_2(t_0)=0.$$

3 Continuation of circular keplerian orbits

First we will analyze the continuation of the circular keplerian orbits of the problem associated the hamiltonian function (7). We will assume that the function W_1 possesses the symmetries S_1 and S_2 and now, we will characterize the circular orbits of the Kepler problem in Poincaré-Delaunay variables. Let $\varphi_{kep}(t,\mathbf{Y}_0) = (Q_1^{(0)}(t,\mathbf{Y}_0), Q_2^{(0)}(t,\mathbf{Y}_0), P_1^{(0)}(t,\mathbf{Y}_0), P_2^{(0)}(t,\mathbf{Y}_0))$ be a solution of the Kepler problem in Poincaré-Delaunay (12) coordinates with initial condition $\mathbf{Y}_0 = ((n_1+1/2)\pi, 0, s^{-1/3}, 0)$ on the y-axis. According to Lemma 2.2, a necessary condition to cut the y-axis orthogonally at the instant $T/4$ are the following periodicity equations:

$$\begin{aligned} Q_1^{(0)}(T/4,\mathbf{Y}_0) &= s\tfrac{T}{4} + n_1\pi = (n_1 + \tfrac{m}{2})\pi, \\ Q_2^{(0)}(T/4,\mathbf{Y}_0) &= 0. \end{aligned}$$

Thus, we obtain a doubly-symmetric circular orbit whose period is $T = 2\pi m/s$, $m \in \mathbb{N}$ with $s \in \mathbb{R}^+$.

Theorem 3.1 *Assume that the function W_1 possesses the symmetries S_1 and S_2 . Let $\varphi_{kep}(t,\cdot)$ be a circular orbit of (14) with $\epsilon = 0$. Then,*

(i) For ϵ sufficiently small, there are initial conditions $\mathbf{Y}_\epsilon = \mathbf{Y}_0 + \mathbf{Y}(\epsilon)$ with $\mathbf{Y}(\epsilon) = (0, \Delta Q_2(\epsilon), \Delta P_1(\epsilon), 0)$ such that $\varphi(t,\mathbf{Y}_\epsilon;\epsilon) = \varphi_{kep}(t,\mathbf{Y}_\epsilon) + \mathcal{O}(\epsilon^{\alpha+i})$ is a doubly-symmetric periodic solution of (14) with period $T = 4\pi m/s$.

(ii) There are initial conditions $\mathbf{Y}_{\Delta Q_2,\epsilon} = \mathbf{Y}_0 + \mathbf{Y}(\Delta Q_2,\epsilon)$ with $\mathbf{Y}(\Delta Q_2,\epsilon) = (0, \Delta Q_2, \Delta P_1(\Delta Q_2,\epsilon), 0)$, where ϵ and ΔQ_2 are sufficiently small, such that $\varphi(t,\mathbf{Y}_{\Delta Q_2,\epsilon};\epsilon) = \varphi_{kep}(t,\mathbf{Y}_{\Delta Q_2,\epsilon}) + \mathcal{O}(\epsilon^{\alpha+i})$ is a doubly-symmetric periodic solution of (14) with period $\tau(\Delta Q_2,\epsilon)$ close to T.

Remark 3.1 *In the above theorem we are using the notation* $\mathbf{Y} = ((n_1 + 1/2)\pi, \Delta Q_2, s^{-1/3} + \Delta P_1, 0) := (\Delta Q_2, \Delta P_1)$.

Proof: Let $\varphi_{kep}(t,\mathbf{Y})$ be a circular Keplerian orbit, with initial condition $\mathbf{Y} = (n_1 + 1/2)\pi, \Delta Q_2, s^{-1/3} + \Delta P_1, 0)$ on the y-axis and in a neighborhood of $\mathbf{Y}_0 = (n_1 + 1/2)\pi, 0, s^{-1/3}, 0)$. By differentiability of the flow with respect to ϵ we have that $\varphi(t,\mathbf{Y};\epsilon) = \varphi_{kep}(t,\mathbf{Y}) + \mathcal{O}(\epsilon^{\alpha+i})$ is the expression of the solutions of the perturbed problem (14). This solution will be doubly-symmetric if at $t = T/4$ it cuts the x-axis orthogonally, and follows from Lemma 2.2, that the periodicity equations must be satisfied, i.e., at $t = T/4$,

$$\begin{aligned} f_1(t,\mathbf{Y},\epsilon) = {} & \left(s^{-1/3} + \Delta P_1\right)^{-3} t - m\pi \\ & + \mathcal{O}(\epsilon^{\alpha+i}) = 0, \\ f_2(t,\mathbf{Y},\epsilon) = {} & \Delta Q_2 + \mathcal{O}(\epsilon^{\alpha+i}) = 0. \end{aligned}$$

Observe that $f_1(T/4,\mathbf{Y},0)\big|_{\mathbf{Y}=0} = f_2(T/4,\mathbf{Y},0)\big|_{\mathbf{Y}=0} = 0$. It follows from the periodicity equations that

$$\left.\frac{\partial(f_1,f_2)}{\partial(t,\Delta Q_2,\Delta P_1)}\right|_{\substack{t=T/4\\ \mathbf{Y}=\mathbf{0}\\ \epsilon=0}} = \begin{pmatrix} s & 0 & -\frac{3\,s^{4/3}\,T}{4} \\ 0 & 1 & 0 \end{pmatrix}.$$

then, we obtain:
(A) $det\left(\frac{\partial(f_1,f_2)}{\partial(\Delta Q_2,\Delta P_1)}\right) = s \neq 0$ and
(B) $det\left(\frac{\partial(f_1,f_2)}{\partial(t,\Delta Q_2)}\right) = \frac{3s^{4/3}T}{2} \neq 0$.

Applying the Implicit Function Theorem in the cases (A) and (B), we have proved the items (i) and (ii), respectively. ■

Considering the Kepler problem in rotating coordinates written in Poincaré-Delaunay variables (12), The Keplerian circular solutions with initial condition $\mathbf{Y}_0 = ((n_1 + 1/2)\pi, 0, s^{-1/3}, 0)$ on the y-axis is given by $\psi(t,\mathbf{Y}_0;0) = (Q_1(t), Q_2(t), P_1(t), P_2(t)) = ((s-1)t + (n_1+1/2)\pi, 0, s^{-1/3}, 0)$. To satisfy the symmetry condition at $t = T/4$, by Lemma 2.2, we need to solve the following periodicity equations:

$$\begin{aligned} Q_1(T/4,Y_0) &= (s-1)\tfrac{T}{4} + (n_1 + 1/2)\pi \\ &= (n_1 + m)\pi, \\ Q_2(T/4,Y_0) &= 0. \end{aligned}$$

Solving the periodicity equations above we have a doubly-symmetric circular solution of the Kepler problem with period $T/4 = \pi(2m - 1)/2(s-1)$.

Theorem 3.2 *Assume that W_2 is invariant about the symmetries S_1 and S_2 and let $\psi_{kep}(t,\cdot)$ be a circular Keplerian solution of the problem (16) with $\epsilon = 0$.*

(i) Then for ϵ sufficiently small there are initial conditions $\mathbf{Y}_\epsilon = \mathbf{Y}_0 + \mathbf{Y}(\epsilon)$ with $\mathbf{Y}(\epsilon) =$

$(0, 0, \Delta P_1(\epsilon), \Delta P_2(\epsilon))$ *such that* $\psi(t, \mathbf{Y}_\epsilon; \epsilon) = \psi_{kep}(t, \mathbf{Y}_\epsilon) + \mathcal{O}(\epsilon^{\alpha+i})$ *is a doubly-symmetric periodic solution of perturbed problem (16) with period* $T = 2\pi(2m-1)/(s-1)$ *whether* $(s-1) \neq (2m-1)/k$ *for all* $k \in \mathbb{N}$.

(*ii*) *There are initial conditions* $\mathbf{Y}_{\Delta P_1,\epsilon} = \mathbf{Y}_0 + \mathbf{Y}(\Delta P_1, \epsilon)$ *where* $\mathbf{Y}(\Delta P_1, \epsilon) = (0, 0, \Delta P_1, \Delta P_2(\Delta P_1, \epsilon))$, ϵ *and* ΔP_1 *sufficiently small, such that* $\psi(t, \mathbf{Y}_{\Delta P_1,\epsilon}; \epsilon) = \psi_{kep}(t, \mathbf{Y}_{\Delta P_1,\epsilon}) + \mathcal{O}(\epsilon^{\alpha+i})$ *is a doubly-symmetric period solution of the perturbed problem (16) with period* $\tau(\Delta P_1, \epsilon)$ *close to* $T = 2\pi(2m-1)/(s-1)$ *whenever that* $(s-1) \neq (2m-1)/k$ *for all* $k \in \mathbb{N}$.

Remark 3.2 *In the formulation of the previous theorem we are assuming the notation* $\mathbf{Y} = ((n_1 + 1/2)\pi, \Delta Q_2, s^{-1/3} + \Delta P_1, 0) := (\Delta Q_2, \Delta P_1)$.

Proof: Let $\psi_{kep}(t, \mathbf{Y})$ be a circular Keplerian solution with initial condition $\mathbf{Y} = ((n_1 + 1/2)\pi, \Delta Q_2, s^{-1/3} + \Delta P_1, 0)$ on the y-axis in a neighborhood of $\mathbf{Y}_0$. By differentiability of the flow with respect to parameter ϵ, the solution of the perturbed problem (16) is given by $\psi(t, \mathbf{Y}; \epsilon) = \psi_{kep}(t, \mathbf{Y}) + \mathcal{O}(\epsilon^{\alpha+i})$. In order to satisfy the symmetry condition at $t = T/4$, by Lemma 2.2, we need to solve the following periodicity equations:

$$\begin{aligned} f_1(t, \mathbf{Y}, \epsilon) = {} & \left((s^{-1/3} + \Delta P_1)^{-3} - 1\right)t - \\ & (m + 1/2)\pi + \mathcal{O}(\epsilon^{\alpha+i}) = 0 \\ f_2(t, \mathbf{Y}, \epsilon) = {} & \Delta Q_2 \cos t + \mathcal{O}(\epsilon^{\alpha+i}) = 0. \end{aligned}$$

where $m \in \mathbb{N}$, at $t = T/4$. Note that $f_1(T/4, \mathbf{Y}; 0)\big|_{\mathbf{Y}=\mathbf{0}} = 0$ and $f_2(T/4, \mathbf{Y}; 0)\big|_{\mathbf{Y}=\mathbf{0}} = 0$. From the periodicity equations follows that at $t = T/4$, $\mathbf{Y} = \mathbf{Y_0}$ and $\epsilon = 0$,

$$\frac{\partial(f_1, f_2)}{\partial(t, \Delta P_1, \Delta P_2)} = \begin{pmatrix} s-1 & -\frac{3\ s^{\frac{4}{3}} T}{4} & 0 \\ 0 & 0 & \cos(\frac{T}{4}) \end{pmatrix}$$

We have the following possibilities for the obtention of periodic solutions.

(*A*) $det\left(\frac{\partial(f_1, f_2)}{\partial(\Delta P_1, \Delta P_2)}\right)\Big|_{t=T/2, \mathbf{Y}=\mathbf{0}, \epsilon=0} = -3\ s^{4/3} T/4\ \cos(T/4) \neq 0$, because $T/4 \neq (k/2)\pi$ for all $k \in \mathbb{N}$. Then the proof of item (i) follows by the Implicit Function Theorem.

(*B*) $det\left(\frac{\partial(f_1, f_2)}{\partial(t, \Delta P_2)}\right)\Big|_{t=T/2, \mathbf{Y}=\mathbf{0}, \epsilon=0} = (s-1)\cos(T/4) \neq 0$, because $T/4 \neq (k/2)\pi$ for all $k \in \mathbb{N}$ and the proof of item (ii)follows by the Implicit Function Theorem. ■

4 Continuation of ellipitc Keplerian orbits

Let $\psi_{kep}(t, \mathbf{Y}_0)$ be an elliptic solution of the problem (11) with $\epsilon = 0$, giving in Delaunay variable (5), T-periodic and doubly-symmetric with initial condition $\mathbf{Y}_0 = (Q_1^0, Q_2^0, P_1^0, P_2^0) = (n_1\pi, (n_2 + 1/2)\pi, s^{-1/3}, p_2) \in S_2$ where $n_1, n_2 \in \mathbb{N}$, $T = 2(2n+1)\pi$, such that $s = 2m/(2n+1)$ with $m, n \in \mathbb{N}$. Then, this solution is given by: $Q_1^{(0)}(t, \mathbf{Y}_0) = (s-1)t + Q_1^0$, $P_1^{(0)}(t, \mathbf{Y}_0) = P_1^0$, $Q_2^{(0)}(t, \mathbf{Y}_0) = Q_2^0 \cos t + P_2^0 \sin t$, $P_2^{(0)}(t, \mathbf{Y}_0) = -Q_2^0 \sin t + P_2^0 \cos t$.

Theorem 4.1 *Assume that the perturbed function* W_2 *is invariant about the symmetries* S_1 *and* S_2 *and let* $\psi_{kep}(t, \cdot)$ *be and elliptic Keplerian solution of the problem (11) with* $\epsilon = 0$. *Then there are initial conditions* $\mathbf{Y}_{\Delta P_2,\epsilon} = \mathbf{Y}_0 + \mathbf{Y}(\Delta P_2, \epsilon)$ *where* $\mathbf{Y}(\Delta P_2, \epsilon) = (0, 0, \Delta P_1(\Delta P_2, \epsilon), \Delta P_2)$, *with* ϵ *and* ΔP_2 *sufficiently small, such that* $\psi(t, \mathbf{Y}_{\Delta P_2,\epsilon}; \epsilon) = \psi_{kep}(t, \mathbf{Y}_{\Delta P_2,\epsilon}) + \mathcal{O}(\epsilon^{\alpha+i})$ *is a doubly-symmetric periodic solution of the perturbed problem (11) with period* $\tau(\Delta P_2, \epsilon)$ *close to* $T = 2(2n+1)\pi$, *whether* s *had the rational form* $s = 2m/(2n+1)$.

Proof: Let $\psi_{kep}(t, \mathbf{Y})$ be an elliptic solution of the Kepler problem with initial condition $\mathbf{Y} = ((n_1\pi, (n_2 + 1/2)\pi, s^{-1/3} + \Delta P_1, p_2 + \Delta P_2)$ on the y-axis in a neighborhood of $\mathbf{Y}_0$ and let $\psi(t, \mathbf{Y}, \epsilon)$, (for $\epsilon \neq 0$ sufficiently small) be a solution of the perturbed problem (11). We know that $\psi(t, \mathbf{Y}, \epsilon) = \psi_{kep}(t, \mathbf{Y}) + \mathcal{O}(\epsilon^{\alpha+i})$ so, in order to satisfy the conditions of symmetries at $t = T/4$, by Lemma 2.1, it is necessary to solve the following periodicity equations:

$$\begin{aligned} f_1(t, \mathbf{Y}, \epsilon) = {} & (s^{-1/3} + \Delta P_1)^{-3} t - m\pi + \\ & \mathcal{O}(\epsilon^{\alpha+i}) = 0, \\ f_2(t, \mathbf{Y}, \epsilon) = {} & -t + (n + 1/2)\pi + \mathcal{O}(\epsilon^{\alpha+i}) = 0, \end{aligned}$$

at $t = T/4$, where $m, n \in \mathbb{R}$. Note that $f_1(T/4, \mathbf{Y}; 0)\big|_{\mathbf{Y}=\mathbf{0}} = 0$, $f_2(T/4, \mathbf{Y}; 0)\big|_{\mathbf{Y}=\mathbf{0}} = 0$. It follows from the periodicity equations that:

$$\frac{\partial(f_1, f_2)}{\partial(t, \Delta P_1, \Delta P_2)}\Bigg|_{\substack{t = \frac{T}{4} \\ \mathbf{Y} = \mathbf{0} \\ \epsilon = 0}} = \begin{pmatrix} s & -\frac{3\ s^{\frac{4}{3}} T}{4} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Note that $det\left(\frac{\partial(f_1,f_2)}{\partial(t,\Delta P_1)}\right)\Big|_{t=T/4,\mathbf{Y}=\mathbf{0},\epsilon=0} = 3\ s^{4/3}\ T/4$. Then by the Implicit Function Theorem, there are analytic functions $\Delta P_1 = \Delta P_1(\Delta P_2, \epsilon)$ and $\tau = \tau(\Delta P_2, \epsilon)$ defined for $|\Delta P_2| < \delta$ and $|\epsilon| < \epsilon_0$ where δ and ϵ_0 are sufficiently small, such that $\Delta P_1(0,0) = 0$, $\tau(0,0) = T$. Then, the doubly-symmetric solution of the perturbed problem with initial condition $\mathbf{Y}_{\Delta P_2,\epsilon} = (n_1\pi, (n_2 + 1/2)\pi, s^{-1/3} + \Delta P_1(\Delta P_2, \epsilon), p_2 + \Delta P_2)$ is given by $\psi(t, \mathbf{Y}_{\Delta P_2;\epsilon}; \epsilon) = \psi_{kep}(t, \mathbf{Y}_{\Delta P_1,\epsilon}) + \mathcal{O}(\epsilon^{\alpha+i})$. This solution is τ-periodic where $\tau(\Delta P_2, \epsilon)$ is close to $T = 2(2n+1)\pi$, whenever s had the rational form $s = 2m/(2n+1)$. ∎

5 New families of doubly-symmetric periodic solutions on the circle problem

We are interested in the study of movement in $\mathbb{R}^2$ of a particle P under the influence of the gravitational force induced by a fixed homogeneous circle problem $\mathcal{C}$. In the literature, we find several authors who study the existence of periodic solutions in the circle problem, we cite [1], [3] e [4]. But, we will prove the existence of some types of doubly-symmetric periodic solutions of the circle problem, in the planar case, that does not appear on the literature.

Considere the homogeneous circle in the (x, y)-plane in Euclidean Three-space $\mathbb{R}^3$, with radius ϵ, and with constant mass density λ and mass $M = 2\pi\lambda\epsilon$. In the cartesian coordinates, the Hamiltonian function of the circle problem is given by

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\|\mathbf{p}\|^2 + V(\mathbf{q}) \qquad (18)$$

where $V(P) = -\frac{M}{2\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{x^2+y^2+z^2+\epsilon^2-2x\epsilon\cos\theta-2y\epsilon\sin\theta}}$ is the potential function associated the problem.

Lemma 5.1 *The potential V associated to the circle problem is invariant under the reflections with respect to x-axis and $y-axis$.*

Proof: Its is clear from definition of V. ∎

In order to take into account the dependence the potential with respect to M, ϵ, and r we denote de potencial as $V(\mathbf{q}, \epsilon, M) = -\frac{M}{2\pi}\int_0^{2\pi} \frac{d\theta}{\|\mathbf{q}-\epsilon e^{i\theta}\|}$ where $\epsilon e^{i\theta} = (\epsilon\cos(\theta), \epsilon\sin(\theta), 0)$. The potential $V(\mathbf{q}, \epsilon, M)$ of the homogeneous circle problem satisfies:

(1) $V(\mathbf{q}, \epsilon, cM) = c\ V(\mathbf{q}, a, M)$.
(2) $\nabla V(\mathbf{q}, \epsilon, cM) = c\ \nabla V(\mathbf{q}, \epsilon, M)$.
(3) $V(c\mathbf{q}, c\epsilon, M) = c^{-1}\ V(\mathbf{q}, \epsilon, M)$.
(4) $\nabla V(c\mathbf{q}, c\epsilon, M) = c^{-2}\ \nabla V(\mathbf{q}, \epsilon, M)$.

(for more details see [3]). Therefore we obtain that if $\mathbf{q}(t)$ is a solution of the problem $\ddot{\mathbf{q}}(t) = -\nabla V(\mathbf{q}, \epsilon, M)$ and let $\alpha, \beta \in \mathbb{R}$ such that $\alpha^3\beta^2 = \frac{\tilde{M}}{M}$. Then, $\mathbf{s}(t) = \alpha\mathbf{q}(\beta t)$ is a solution of the problem $\ddot{\mathbf{s}}(t) = -\nabla V(\mathbf{s}, \alpha\epsilon, \tilde{M})$.

Remark 5.1 *Follows by the previous comments that if we obtain the solution for the problem with radius ϵ we can obtain the solution for the problem with any radius.*

We can to prove that the Hamiltonian function of the circle problem ca be writing as

$$H(\mathbf{q}, \mathbf{p}, \epsilon) = \frac{\|\mathbf{p}\|^2}{2} - \frac{M}{\|\mathbf{q}\|} + \epsilon^2 \frac{M}{4\|\mathbf{q}\|^3} + \mathcal{O}(\epsilon^4) \qquad (19)$$

where ϵ is the radius of the circle, $\mathbf{q} = (x, y)$, $\mathbf{p} = (p_x, p_y)$ and $\cos\vartheta = z/\|\mathbf{q}\|$ (for more details see [2]).

If we define $W_1(\mathbf{q}, \epsilon) = \frac{M}{4\|\mathbf{q}\|^3} + \mathcal{O}(\epsilon^4)$, follows from Lemma 5.1 that the function W_1 is invariant about reflections with respect to x-axis and y-axis. Then the Hamiltonian function (19) can be writing in the form

$$H(\mathbf{q}, \mathbf{p}, \epsilon) = \frac{\|\mathbf{p}\|^2}{2} - \frac{M}{\|\mathbf{q}\|} + \epsilon^2 W_1(\mathbf{q}, \epsilon) \qquad (20)$$

and we have a particular case of the problem (1). The Hamiltonian function (20) in the Poincaré-Delaunay variables is given by

$$H(\mathbf{Q}, \mathbf{P}, \epsilon) = -\frac{1}{2P_1^2} + \epsilon^2 W_1(Q, P, \epsilon).$$

Since the circle problem is invariant by the rotations around the z-axis, we write the problem in the rotating coordinates. Therefore, the Hamiltonian function of the problem is

$$\begin{aligned} K(\mathbf{q}, \mathbf{p}, \epsilon) \;&= \frac{\|\mathbf{p}\|^2}{2} - \frac{1}{\|\mathbf{q}\|} - (xp_y - yp_x) + \\ &\epsilon^2 K_1(\mathbf{q}) + \mathcal{O}(\epsilon^4), \epsilon), \end{aligned}$$

where K_1 is as defined before. If $W_2(q, \epsilon) = K_1(\mathbf{q}) + \mathcal{O}(\epsilon^4)$ the problem is as in (2) with $\alpha = 0$.

5.1 Doubly-symmetric periodic solutions of the circle problem near of of circles

Applying the results of the section 3 we obtain families of doubly-symmetric periodic solutions of the circle problem for any value of the radius ϵ. Note that by the Theorems 3.1 and 3.2 the parameter ϵ is sufficiently small but, by Remark 5.1, the results can be extend to every ϵ.

Theorem 5.1 *Considere the planar circle problem with radius ϵ. Let $m, p, q \in \mathbb{N}$ and $s \in \mathbb{R}^+$ is chosen conveniently in order to avoid singularities. Then for every positive value of the parameter ϵ we obtain:*

$i)-$ There are one parameter family, that depends on ϵ, for the infinitesimal particle such that its motion is a doubly-symmetric $2\pi m/s$-periodic solution near a Keplerian circular orbit with radius $s^{-2/3}$ and period T.

$ii)-$ There are one parameter family, that depends on ϵ, for the infinitesimal particle such that its motion is a doubly-symmetric $2\pi(2m-1)/(s-1)$-periodic solution , since that $\frac{2m-1}{s-1} \notin \mathbb{N}$, near a Keplerian circular orbit with radius $s^{-2/3}$ and period T.

$iii)-$ There are two parameters family for the infinitesimal particle such that its motion is a doubly-symmetric τ-periodic solution near a Keplerian circular orbit with radius $s^{-2/3}$ and period $T = 2\pi m/s$.

$iv)-$ There are two parameters family, that depends on ϵ, for the infinitesimal particle such that its motion is a doubly-symmetric τ-periodic solution ($\tau = 2\pi p/q$) , since that $\frac{2m-1}{s-1} \notin \mathbb{N}$, near a Keplerian circular orbit with radius $s^{-2/3}$ and period $T = 2\pi(2m-1)/(s-1)$.

5.2 Doubly-symmetric periodic solutions of the circle problem near of elipse

Using the Theorem 4.1 we prove the following result for the planar circle problem.

Theorem 5.2 *Considere the planar circle problem with radius ϵ. Let $m, p, q \in \mathbb{N}$ and $s \in \mathbb{R}^+$ is chosen conveniently in order to avoid singularities. Then for every value of the parameter ϵ, there are one parameter family, that depends on ϵ for the infinitesimal particle such that its motion is a doubly-symmetric τ-periodic solution , $\tau = 2\pi p/q$, since that $m/s = n + 1/2$ for $n, m \in \mathbb{N}$, near a Keplerian elliptic orbit with period $T = 2\pi m/s$.*

6 Conclusion

We observe that in theorems 3.1, 3.2 and 4.1 we obtain planar doubly-symmetric periodic solutions for any analytical perturbation of Kepler problem (in retangular ou rotating coordinates). In other words, the unique restriction under the perturbed function W_1 (or W_2) to continue the periodic solutions is the analyticity. In Celestial Mechanics there exist a great class of problems who are analytic perturbations of the Kepler problem (for exemple, see [2]). Therefore, the results in this work can be applied to many others problems in Celestial Mechanics.

In a next stage, we are going to consider analytical perturbations of the Kepler problem in the spatial case, and we use the spatial version of Delaunay and Poincaré-Delaunay variables. In this case others symmetries need to be considered.

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