Sierpinski curve for total ordering of a graph-based adaptive simplicial-mesh refinement for Finite Volume discretizations

Sanderson L. Gonzaga de Oliveira
Mauricio Kichinhevsky
Instituto de Computação, UFF
24210-240, Niterói, RJ
E-mails: {sgonzaga,kisch}@ic.uff.br

Abstract: The solution of partial differential equations with finite differences, volumes or elements produces linear (or non-linear) systems. Nodes must be numbered so that linear systems have each row corresponding to a specific node. The ordering of all nodes is a step which can be carried out in several ways. In this work, numbering is performed by a modified Sierpinski-like covering. We apply an adaptive mesh refinement (AMR) based on the Finite Volume method (FVM) in order to numerically solve partial differential equations (PDEs). The mesh is represented by a graph data structure and parent nodes in the AMR are not stored. This scheme follows the Autonomous Leaves Graph (ALG) concepts and intends to reduce the computational cost to construct the refined simplicial-mesh in evolutionary problems in relation to other similar works. Numerical solution of the heat conduction equation demonstrates the efficiency and advantages of this scheme.

1. Introduction

Numerical solution of PDEs may require the use of a mesh refinement strategy that concentrates more mesh points where the solution and/or its derivatives change rapidly. For time-dependent problems, adaptive methods become particularly important since, by the dynamic nature of such problems, may happen migration (or occurrence) of regions that have rapid solution change. Thus, this work follows the concept of a graph data structure proposed by Burgarelli, Kischinhevsky and Biezuner (2006), named Autonomous Leaves Graph. The original ALG numerically solve PDEs using a mesh composed by square-shaped control volumes and the initial discrete domain is, generally, a unit square.

One may consider which shape each element of the mesh has. Thus, triangular meshes, or triangulations, are one of the most widely used representations for geometric models. A triangulation is a 2D simplicial complex, a simple structure with convenient combinatorial properties (Velho, Figueiredo and Gomes, 1999). An non-uniform mesh comprised of triangular control volumes can be more appropriated near physical boundary regions and features of a problem with complex geometries. Therefore, this technique operates all the required elements for the AMR in the Finite Volume method. A simplicial mesh is shown in a experimental test in order to exemplify this technique.

Thus, this work uses a discrete adaptive mesh comprised by triangular control volumes following the ALG technique (Gonzaga and Kischinhevsky, 2008). The main novelty of this work is to adapt a well-known space-filling curve in order to number the nodes. The Sierpinski curve (Sierpinski, 1912, 1916) was proposed by Waclaw Franciszek Sierpinski (1882-1969) in 1912 and complemented in 1916. It is widely used, for example, in fractals. Since the Sierpinski curve was proposed to link simplicial meshes uniformly refined, we adapted the algorithm to number the AMR in each time-step of the problem.

After this brief introduction, Section 2 treats the graph data structure. Afterwards, Section 3 describes the adapted Sierpinski space-filling curve and mesh total ordering. Subsequently, Section 4 discusses preliminary experimental tests. Finally, Section 5 draws some considerations.

2. Graph-based AMR technique with triangular control volumes

Velho, Figueiredo and Gomes (1999) described a dual graph for simplicial mesh to build triangle strips by heuristics. Those authors considered several issues related to paths on the dual graph of general simplicial meshes. Following the concepts presented by Burgarelli, Kischinhevsky and Biezuner (2006), this present work uses a graph-based adaptive simplicial-mesh refinement that furnishes all the requirements for solution of PDEs based on the FVM with triangular control volumes intending to be competitive with other AMR approaches.

ALG explicitly gives all the relations among control volumes. The nodes correspond to the control volumes and two nodes are connected if their associated volumes have a common edge. Thus, when a control volume is refined, a parent control volume node is substituted by a new sub-graph, i.e., a pack with four control volume nodes and three transition nodes. Transition nodes indicate the refinement level of the control volume in relation to their neighbor control volumes. The refinement process is shown in Figure 1, whose most right sub-graph is a pack that regenerates its parent control volume node sketched on its left-hand side.

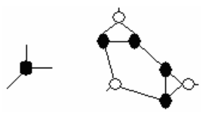
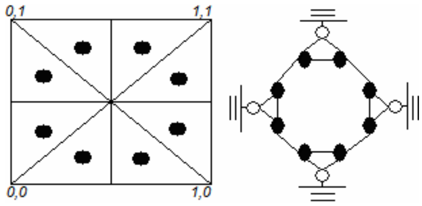


Figure 1: Right-hand side sub-graph created after a refinement of the left-hand side parent control volume node (Gonzaga and Kischinhevsky, 2008).

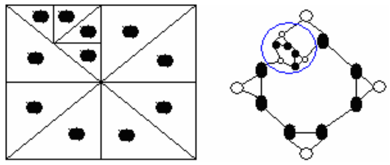
Only four control volume graph nodes that represent the four new control volumes and the three required transition graph nodes are stored in the new pack. Since parent nodes are deleted in the local refinement of each triangular control volume, children control volume nodes become autonomous and need to be renumbered.

Figure 2a depicts an example of an initial discretization represented by an unit square. Figure 2b depicts a graph that represents this initial discretization. The barycenter of the triangle is depicted by a black point in Figure 2a. Control volume graph nodes in Figure 2b represent each control volume of Figure 2a. Graph links in Figure 2b are represented by lines. Figure 3 represents a control volume from Figure 2a refined and boundaries of the domain are neglected for convenience.



(a) (b)

Figure 2: (a) Unit square as the problem domain; (b) links for the graph data structure – nodes representing the refinement level 0 (Gonzaga and Kischinhevsky, 2008).



(a) (b)

Figure 3: (a) Refinement example with triangular control volumes; (b) graph with transition and control volume graph nodes that forms the scheme of triangular mesh control volume refinement (Gonzaga and Kischinhevsky, 2008).

3. Sierpinski-like space-filling curve for the total order relation on the triangular mesh control volumes

A triangular sequence designates a sequential triangulation as well as its generalizations. A triangular sequence defines a total order relation on the set of triangles of a mesh. It is always possible to divide a simplicial mesh in a collection of sub-triangulations so that each new pack is a triangular sequence.

Velho, Figueiredo and Gomes (1999) explained that a generalized sequential triangulation, or Hamiltonian triangulation (its name is a tribute to Willian Rowan Hamilton, 1805-1865) is a triangulation in which there is an ordering $T_1, \ldots, T_N$ of all its triangles so that two consecutive triangles T_j and T_{j+1} share an edge. Thus, a path in a graph is said to be Hamiltonian if it visits all nodes in the graph exactly once. They also claimed that each triangle in a Hamiltonian triangulation has an entry and an exit edges with respect to the Hamiltonian ordering. The knowledge of these edges completely characterizes the sequence. Geometrically, a generalized triangular sequence can be represented by drawing an oriented path on the mesh domain that visits each triangle crossing its entry and exit edges, called a sequential path. Likewise, the scheme proposed here always generates Hamiltonian sequences.

Mesh nodes must be numbered so that the corresponding linear system has each row matching to a specific node. Thus, this work proposes a modified Sierpinski space-filling curve for total ordering of a graph-based simplicial-mesh. In other words, this work proposes a non-regular Sierpinski curve to number the control volumes of the mesh. It is generated by a simple linked list. Therefore, in the sense of data structures, this adaptive refinement scheme by triangular control volumes permits straightforward update of the linked list in the modification of control volume nodes in the adaptive refinement process during PDE evolution.

As examples, Figure 4 sketches successive adaptive refinements in a quadrangular domain with triangular control volumes ordered by the adapted Sierpinski-like curve.

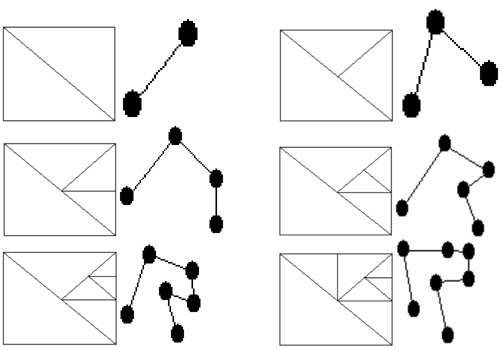


Figure 4: Successive adaptive refinements with control volumes ordered by the adapted Sierpinski-like curve on the right-hand side of each discretization (on its left-hand side).

4. Experimental test

The experimental test consists in the numerical solution of the heat conduction equation, which is a parabolical and evolutionary problem. Consider the two-dimensional heat conduction equation on problem (1).

$$\phi_t = \nabla^2 \phi \qquad (1)$$

$$\phi(u,0) = f(u), u \in \Omega, f(u) \equiv 0$$

$$\phi(u,t) = g(u), u \in \Omega, t \geq 0$$

where $\Omega \subset \Re^2$, thus, $u=(x,y)$. Function f is a smooth function and limited in Ω.

In 1902, Jacques Salomon Hadamard (1865-1963) claimed that, for a mathematical problem correspond to reality, the following basic conditions should be satisfied: the solution must exist (existence); the solution must be determined by data of a unique form (unicity); the solution must depend on data of continual form (stability). Thus, problem (1) is well-defined in the sense of Hadamard. Therefore, consider the numerical solution of (1) in a unit square. Consider now an implicit formulation, thus, Equation (2) follows the FVM basic formulation for irregular meshes.

$$\frac{M\phi_p^{n+1}}{\Delta t} - \nabla^2 \phi_p^{n+1} = \frac{M\phi_p^n}{\Delta t} \qquad (2)$$

where M represents the area of triangle p, and $\Delta t=0.1$.

We chose a problem with boundary conditions at the top, at the bottom and on the left sides of the unit square with a unique prescribed boundary value f and the right side of the domain with a different value. Figure 5 illustrates discretizations in different refinement levels: 1, 3 and 5, respectively. Each discretization corresponds for the respecttive time step. In this example, the refinement criterion is that the difference between the absolute values of neighboring control volumes must be greater than 1.0, thus, both control volumes are refined. Furthermore, boundary conditions at the top, at the bottom and on the left sides of the unit square have a unique prescribed boundary value f and the right side has a different value: the initial domain and discretizations of 1, 3 and 5 refinement levels with 32, 386, and 1148 control volumes, respectively.

In relation to the test performed, this scheme resulted in a linear system with symmetric and positive definite coefficient matrix. However, this scheme can be employed in both symmetric and non-symmetric matrices. Moreover, the Gradient Conjugate method (Hestenes and Stiefel, 1952) is employed in this work, which is one of the most promient iterative method for solving sparse systems of linear equations.

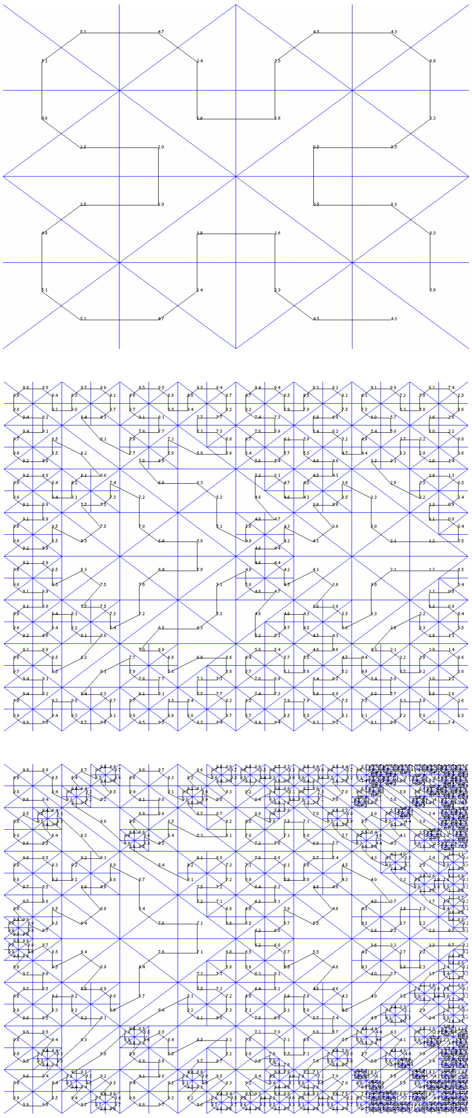


Figure 5: An adaptive refinement numerical solution of the heat conduction equation.

5. Considerations

This work proposes a total ordering of simplicial-meshes by a modified Sierpinski-like covering. In other words, we apply a modified Sierpinski-like space-filling curve for total ordering in an AMR based on the Finite Volume method (FVM) in order to solve PDEs. This process is as simple as the original ALG technique.

This scheme proposed here pursues to have flexibility in ordering the mesh with control volumes refined in different levels simply updating the linked list in the adaptive refinement process.

The mesh total ordering scheme proposed here is an adapted Sierpinski-like curve, which is generated through a simple linked list. In other words, it is a non-regular version of the well-known space-filling curve. The refinement scheme of the triangular control volume allows straightforward update of the linked list for mesh total ordering in the sense of data structures. More experimental tests should be provided in order to demonstrate that this modified Sierpinski-like curve in a local adaptive refinement is more efficient in the process of update of the linked list to number the mesh in comparison to the original ALG technique, which uses the modified Hilbert curve (proposed by David Hilbert (1862-1943) in 1891). Tests may demonstrate that the modified Hilbert curve is more computationally expensive than the Sierpinski curve applied here because the inerent concepts of both space-filling curves. Namely, since the nodes are not static because the AMR, the modified Hilbert curve has inerent conditions whereas the update of the modified Sierpinski-like curve applied here is straighforward in the sense of a simple linked list data structure.

Since the scheme proposed here furnishes a Hamiltonian sequence for total ordering, it is a triangular strip-like scheme. Such sequential structure presents reduced connectivity information. Therefore, future works may demonstrate that this scheme is appropriated to be used in algorithms that generate triangle strip for accelerated rendering, algorithms that compute sequential triangulations for geometry compression and also in theoretical investigation of paths on simplicial meshes, such as those described by Velho, Figueiredo and Gomes (1999). The relative gain in efficiency may be even more compelling for three-dimensional problems.

The presented test show a preliminary result. This work should continue in order to develop a computational implementation of other problems. Those tests may demonstrate the efficiency of this scheme and also its advantages in comparison to others applied in AMR techniques.

Acknowledgments

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