Riemann Surfaces Obtained from Hyperbolic Tessellations $\{10\lambda, 2\lambda\}$

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1 Introduction

Fuchsian groups are discrete subgroups of the projective special linear group consisting of 2×2 matrices whose elements are in $\mathbb{R}$, that is, $PSL(2, \mathbb{R})$, or equivalently, the set of Möbius transformations, [4]. The elements of $PSL(2, \mathbb{R})$ are isometries which act by homeomorphisms on the upper-half plane $\mathbb{H}^2 = \{z \in \mathbb{C} : Im\,(z) > 0\}$.

The main purpose of this paper is to construct Riemann surfaces from hyperbolic tessellations in the form $\{10\lambda, 2\lambda\}$, where λ is a natural number, $\lambda > 2$. The process that we will use consists of a classic form, that is, through the space quotient with respect to a discontinuous group action on a spacr topological space. In particular, if Γ is a finite co-area Fuchsian group acting on $\mathbb{H}^2$, then it is possible to endow the quotient space $\mathbb{H}^2/\Gamma$ (the space of the Γ-orbits) with a structure of a Riemann surface, [3].

In this direction, the the crucial problem is to establish the necessary conditions on $\{10\lambda, 2\lambda\}$, so that we obtain discrete groups $\Gamma_{10\lambda}$ and, hence, Fuchsian groups that correspond to such tessellations. The first step is to consider, for each λ, a pairings set for the edges of the regular hyperbolic polygon $P_{10\lambda}$ (fundamental region of $\Gamma_{10\lambda}$) associated with the tesselltion $\{10\lambda, 2\lambda\}$, Satisfying, of course, conditions previously established. Consequently, the group $\Gamma_{10\lambda}$ can be obtained, once your generators are exactly the hyperbolic transformations that realize those pairings. Under these conditions, the hyperbolic area of $\mathbb{H}^2/\Gamma$ is $\mu\left(\mathbb{H}^2/\Gamma\right) = 4\pi\,(g-1)$, for each g in consideration, where g is the genus of a orientable surface compact $\mathbb{H}^2/\Gamma$, $\Gamma \simeq \Gamma_{10\lambda}$. Therefore, for each λ, the group Γ_{4g} is geometrically finite, [2].

This paper is organized as follows. In Section 2 we present some basic considerations on the models of the hyperbolic plane and on Fuchsian groups for the development of this paper. We also present the Theorem 2.1 on Riemann surfaces, as well as the model of the fundamental region (geometry of the surface) considered in the construction of the discrete groups. Section 3 contains the basic results on hyperbolic tessellations $\{p, q\}$ and discrete groups Γ_p obtained from $\{p, q\}$. Finally, in Section 4 we consider the groups $\Gamma_{10\lambda}$ associated with the hyperbolic tessellations $\{10\lambda, 2\lambda\}$.

2 Preliminaries

We consider the upper-half plane $\mathbb{H}^2 = \{z \in \mathbb{C} : Im\,(z) > 0\}$ equipped with a Riemannian metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

With this metric, $\mathbb{H}^2$ is a model of the hyperbolic plane. Let $PSL(2,\mathbb{R})$ be the set of all Möbius transformations, $T : \mathbb{C} \to \mathbb{C}$, given by $T(z) = \frac{az+b}{cz+d}$, where $a,b,c,d \in \mathbb{R}$ and $ad - bc = 1$. To this transformation the following pair of matrices are associated

$$A_T = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence, $PSL(2,\mathbb{R}) \simeq SL(2,\mathbb{R})/\{\pm I_2\}$, where $SL(2,\mathbb{R})$ is the group of real matrices with determinant equal to 1 and I_2 denotes the 2×2 identity matrix.

A Fuchsian group Γ is a discrete subgroup of $PSL(2,\mathbb{R})$, that is , $\Gamma < PSL(2,\mathbb{R})$ consists of orientation preserving isometries $T : \mathbb{H}^2 \to \mathbb{H}^2$, acting on $\mathbb{H}^2$ by homeomorphisms. Além disso, Γ é um grupo Fuchsiano se, e somente se Γ acts properly discontinuously on $\mathbb{H}^2$, [4] and [1].

Poincaré disc $\mathbb{D}^2$, that is, $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$ equipped with a Riemannian metric

$$ds^2 = \frac{4(dx^2 + dy^2)}{[1 - (x^2 + y^2)]^2},$$

where $z = x + Im \cdot y$ and Im is an imaginary number, it is other model of hyperbolic plan. similarly, a discrete group[1] Γ_p of orientation preserving isometries $T : \mathbb{D}^2 \to \mathbb{D}^2$ will also be called of grupo Fuchsiano. Thus, if $T_1 \in \Gamma_p < PSL(2,\mathbb{C})$, then

$$T_1(z) = \frac{az + b}{\bar{b}z + \bar{a}}, \quad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1.$$

Moreover, we can write $T_1 = f \circ T \circ f^{-1}$, where $T \in PSL(2,\mathbb{R})$ and $f : \mathbb{H}^2 \to \mathbb{D}^2$ is an isometry given for, [1],

$$f(z) = \frac{zi + 1}{z + i}.$$

[1]In this paper, Γ and Γ_p will represent discreet groups of isometries acting on $\mathcal{H}^2$ and $\mathbb{D}^2$, respectively and $\Gamma \simeq \Gamma_p$. Moreover, $\mathbb{H}^2/\Gamma$ and $\mathbb{D}^2/\Gamma_p$ represent the same surface (g-torus).

Associated with a Fuchsian group Γ there is a fundamental region P^2, resulting from the action of Γ on $\mathbb{H}^2$. This fundamental region has a polygonal shape containing 10λ edges. Therefore, we may endow the quotient space $\mathbb{H}^2/\Gamma$ with a metric of a Riemann surface, this is, a topological space which, when viewed locally, is essentially the same as the complex plane. As it is well known, every compact Riemann surface with genus $g \geq 2$ may be modeled in the hyperbolic plane, [2]. The pairings of the 10λ edges of a regular hyperbolic polygon $P_{10\lambda}$, to be considered in Section 4, leads to an oriented compact surface $\mathbb{H}^2/\Gamma$, with genus g, where $\Gamma \simeq \Gamma_{10\lambda}$ is the Fuchsian group associated with a hyperbolic tessellation $\{10\lambda, 2\lambda\}$. Therefore, for each g, the Fuchsian group is co-compact, [1]. Hence, the hyperbolic area[3] of $P_{10\lambda}$, $\mu\,(P_{10\lambda}) = \mu\left(\mathbb{H}^2/\Gamma\right)$, is finite. Consequently, the group $\Gamma_{10\lambda}$ contains no parabolic elements, that is, Möbius transformations $T \in PSL(2,\mathbb{R})$ such that their traces are equal to 2, $tr(T) = |a + d| = 2$. We also assume that the group $\Gamma_{10\lambda}$ contains no elliptic elements, $T \in PSL(2,\mathbb{R})$ such that $tr(T) = |a + d| < 2$, this assumption is sufficient for the projection $\pi : \mathbb{H}^2 \to \mathbb{H}^2/\Gamma$ to be a covering application, [2]. Hence, $\Gamma_{10\lambda}$ contains only hyperbolic elements, $T \in PSL(2,\mathbb{R})$ such that $tr(T) = |a + d| > 2$. Consequently, the hyperbolic area of $\mathbb{H}^2/\Gamma$is given by $\mu\left(\mathbb{H}^2/\Gamma\right) = 4\pi(g - 1)$, where g is the genus of the surface $\mathbb{H}^2/\Gamma$, [1] and [4] .

If Γ is a Fuchsian group, then for each $x \in \mathbb{H}$, the set

$$\Gamma(x) = \{T(x) : T \in \Gamma\}$$

is called Γ -*orbit* of the point x. And the subgroup of Γ,

$$\Gamma_x = \{T \in G : T(x) = x\},$$

is called *stabilizer of* x in Γ.

Quotient space $\mathbb{H}^2/\Gamma$ is constructed through following equivalence relations on $\mathbb{H}^2$:

$$z_1 \sim z_2 \Leftrightarrow \exists\ T \in \Gamma \text{ tal que } z_2 = T(z_1). \quad (1)$$

[2]Including the edges of the polygon P and its interior.

[3]A métrica sobre $\mathbb{H}^2/\Gamma$ é induzida da métrica hiperbólica sobre $\mathbb{H}^2$, de modo que $\mathbb{H}^2/\Gamma$ é, de fato, um espaço topológico.

Moreover, the equivalence classof an element $z \in \mathbb{H}^2$, $[z]$, is such that $[z] = \Gamma$-orbit of z. Thus, the elements of the space $\mathbb{H}^2/\Gamma$ are the Γ-orbits, this is,

$$\mathbb{H}^2/\Gamma = \{[z] : z \in \mathbb{H}^2\}.$$

Space $\mathbb{D}^2/\Gamma_p$ is constructed similarly. Topologicaly, any g-torus $\mathcal{T}_g$ locally isometric to $\mathbb{D}^2$ can be obtained by the quotien of $\mathbb{D}^2$ by a Fuchsian group Γ_p, this is,[4] $\mathcal{T}_g = \mathbb{D}^2/\Gamma_p$. Of a general form, for each genus g, the action of the group Γ_p in $\mathbb{D}^2$ can be processed by the identification of the edges of a regular polygon P_p of p edges in $\mathbb{D}^2$ for isometries of $\mathbb{D}^2$ that generate Γ_p.In Section 4 we will consider generalized pairings of edges of hyperbolic polygons $P_{10\lambda}$ (Dirichlet regions for Γ_p) associated with hyperbolic tessellations $\{10\lambda, 2\lambda\}$.

We end up that section with the theorem:

Theorem 2.1 *Let D be a subdomain of $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and let G be a group of Möbius transformations which leaves D invariant and which acts discontinuously[5] in D. Then D/G is a Riemann surface.*

2.1 Fundamental region

The Fuchsian groups $\Gamma_{10\lambda}$ that we will consider are obtained of regular hyperbolic polygons with 10λ edges, fundamentals regions. Furthermore, 2λ other polygons meet at each vertex of $P_{10\lambda}$. Hence, the corresponding tessellations of the hyperbolic plane is denoted by $\{10\lambda, 2\lambda\}$.The regular polygon $P_{10\lambda}$ tessellates the hyperbolic plane $\mathbb{H}^2$. Hence, for every λ, $P_{10\lambda}$ is a fundamental region characterizing the surface $\mathbb{H}^2/\Gamma$, where $\Gamma_{10\lambda} \simeq \Gamma$.

This is the reason we consider in this section the concept of fundamental region of a Fuchsian group. To what follows, $\mathcal{X}$ is a metric space and G is the homeomorphism group acting on $\mathcal{X}$.

Thus, we may now consider the concept of a fundamental region.

Definition 2.1 *Let $\mathcal{X}$ be a metric space, and Γ a group of homeomorphisms acting properly discontinuously on $\mathcal{X}$. A closed subset $P \subset \mathcal{X}$ and $P \neq \emptyset$, is called a* fundamental region *of* Γ *if:*

1. $\bigcup_{T\in\Gamma} T(P) = \mathcal{X}$
2. $\overset{\circ}{P} \cap T(\overset{\circ}{P}) = \emptyset$ *for all* $T \in \Gamma - \{Id\}$.

The set $\overset{\circ}{P}$ denotes the interior of P. The family $\{T(P) : T \in \Gamma\}$ is called a tessellation of $\mathcal{X}$.

Definition 2.2 *Let $T(z) = \frac{az+b}{cz+d} \in PSL(2,\mathbb{R})$ be an isometry of $\mathbb{H}^2$, with $c \neq 0$. The set*

$$C_T = \{z \in \mathbb{C} : |cz + d| = 1\},$$

is called an isometric circle *of T.*

Let Γ be a Fuchsian group whose elements are orientation preserving isometries in the unit disc $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$, that is,

$$T(z) = \frac{az+b}{\bar{b}z+\bar{a}}, \quad |a|^2 - |b|^2 = 1, \quad T \in \Gamma.$$

is also a model of the hyperbolic plane[6]. Consider $\check{I}(T) = \{z \in \mathbb{C} : \left|\bar{b}z + \bar{a}\right| > 1\}$, then $R_0 = \overline{\bigcap_{T\in\Gamma} \check{I}(T) \cap \mathbb{D}^2}$ establishes a Ford region, that is, a fundamental region for Γ, where $\overline{X}$ denotes the closure of the set X, [4]. Thus, R_0 is a fundamental region consisting of the points in $\mathbb{D}^2$ outside of the isometric circles of all the isometries of Γ. We are going consider only regular region of the form R_0.

Let Γ be a Fuchsian group and $z_0 \in \mathbb{H}^2$ such that $T(z_0) \neq z_0$, for all $T \in \Gamma$. We define the *Dirichlet region* (*Domain*) *for* Γ *centered at* z_0 to be the set $\mathcal{D}_{z_0}(\Gamma)$, given for

$$\{z \in \mathbb{H}^2 : d(z, z_0) \leq d(z, T(z_0)), \ \forall\ T \in \Gamma\}.$$

Theorem 2.2 [4] *If Γ is Fuchsian group, then $\mathcal{D}_{z_0}(\Gamma)$ is a fundamental region for Γ. Moreover, $\mathcal{D}_{z_0}(\Gamma)$ is locally finite.*

In this paper we will consider only fundamentais regions R_0 such that $R_0 = \mathcal{D}_{z_0}(\Gamma)$ for for some $z_0 \in \mathbb{H}$.

Let Γ a Fuchsian group fuchsiano cocompact, this is, a discrete group for which $\mu\left(\mathbb{H}^2/\Gamma\right) < \infty$ and without parabolic elements,

[4]Or $\mathcal{T}_g = \mathbb{H}^2/\Gamma$.

[5]See [2].

[6]In this paper, Γ and Γ_p will represent discreet groups of isometries acting on $\mathcal{H}^2$ and $\mathbb{D}^2$, respectively and $\Gamma \simeq \Gamma_p$. Moreover, $\mathbb{H}^2/\Gamma$ and $\mathbb{D}^2/\Gamma_p$ represent the same surface (g-torus).

and $\mathcal{D}_{z_0}(\Gamma)$ a Dirichlet region for Γ. There exists in $\mathcal{D}_{z_0}(\Gamma)$ a finite number of vertexes, let us say r, that are fixed points of elliptic elements of Γ.

Let $m_1, \ldots, m_r$ be the orders of those elliptic elements and g the genus of the compact surface and orientable $\mathbb{H}^2/\Gamma$. In this case, we call $(g; m_1, \ldots, m_r)$ the *signature of* Γ.

If Γ no contains elliptic elements, its signature is $(g; 0, \ldots, 0)$, or simply, $(g; -)$. Thus, we have, [4],

$$\mu(\mathbb{H}^2/\Gamma) = 2\pi \left[(2g-2) + \sum_{k=1}^{r} \left(1 - \frac{1}{m_k} \right) \right].$$

Moreover, if $(g; -)$ is the signature of Γ, then $\mu(\mathbb{H}^2/\Gamma) = 2\pi\left[(2g-2)\right]$. That fact is enough so that we have the space quotient. $\mathbb{H}^2/\Gamma$ (or $\mathbb{D}^2/\Gamma_{10\lambda}$) locally isometric to $\mathbb{H}^2$ (or $\mathbb{D}^2$), [2].

3 Discrete Group Obtained from Regular Hyperbolic Tessellations

According to Theorem 2.1, a Riemann surface can be obtained, for example, when we consider the quotient space $\mathbb{D}^2/\Gamma_p$, where the group Γ_p acts properly discontinuously on $\mathbb{D}^2$, this is, if and only if Γ_p is Fuchsian group and, hence, a discrete group of isometries.

This section has as objective the construction of groups discrete of isometries Γ_p from regular hyperbolic polygons P_p that correspond to the regular hyperbolic tessellations $\{p, q\}$. The construction process will be through obtaining of a pairings set for the edges of P_p, satisfying, of course, conditions previously established. Therefore, we extended the equivalence relation in (1) for the set of vertexes and edges of P_p.

We consider only tessellation on $\mathbb{H}^2$ formed by a region $R_0 = \mathcal{D}_{z_0}(\Gamma)$, this is, Dirichlet tesselltion. We will begin with the concept of regular hyperbolic tessellation.

Definition 3.1 *A* regular tessellation *of the hyperbolic plan is a partition of this plan in isometric regular polygons not put upon, all congruent, subjects to the restriction of if they intercept only in your edges or vertexes, so that to we have the same number of polygons sharing a same vertex, independent of the vertex.*

If the polygons of a tessellation on $\mathbb{H}^2$ contain p edges, where each vertex is covered again by q of those polygons, then the tessellation will be denoted for $\{p, q\}$. In particular, if $p = q$, then the tessellation is called *self-dual*.

As the sum of the internal angles of a hyperbolic triangle is smaller than π, $\{p, q\}$ is a hyperbolic tessellation on $\mathbb{H}^2$ if and only if,

$$\frac{2\pi}{p} + \frac{2\pi}{q} < \pi \Leftrightarrow (p-2)(q-2) > 4.$$

Therefore, there is an infinite number of regular tessellations on $\mathbb{H}^2$.

3.1 Conditions on Pairings for Edges of P_p.

In [5] Vieira *at al* proposed a systematic procedure of construction of Fuchsian groups from self-dual tessellation $\{4g, 4g\}$. In that case, a canonical set of pairings for the edges of the polygon P_{4g} is that in that any two edges τ_i and τ_{i+2} in a cyclical order are such that $T(\tau_i) = \tau_{i+1}$, where T is a hyperbolic isometry in Γ_{4g}.

In the case of a tessellation $\{p, q\}$ with $p \neq q$, the search of a generalized pairings set of for the edges of P_p so that Γ_p be discrete, doesn't follow the pattern used for the tessellations $\{4g, 4g\}$ being, therefore, an amount as complex.

Although our objective is to construct discrete groups $\Gamma_{10\lambda}$ from tessellations $\{10\lambda, 2\lambda\}$, we will make here a very general development for any tessellations $\{p, q\}$, with $p \neq q$.

Let Γ_p be the discrete group obtained from respective pairing, to according Theorems 3.1 and 3.2.

For each tessellation $\{p, q\}$, the polygon P_p constitutes the boundary of the Dirichlet domain $\mathcal{D}_{z_0}(\Gamma_p)$ of Γ_p. Thus, we call too P_p of Dirichlet domain.

Definition 3.2 *Let Γ_p be a discrete group of isometries, and u,$v \in \mathbb{D}^2$. We call u and v* congruent *if they belong to the same Γ_p-orbit.*

We observe that two points in $\mathcal{D}_{z_0}(\Gamma_p)$ may be congruent only if they belong to the boundary of $\mathcal{D}_{z_0}(\Gamma_p)$, once $\mathring{P} \cap T(\mathring{P}) = \emptyset$ for all $T \in \Gamma - \{Id\}$.

Definition 3.3 *Let* $v_1, \ldots, v_p$ *be the* p *vertexes of* P_p. *We call* cycle *the equivalence class obtained from each one of the vertexes, this is, a cicly is a set in the form*

$$C_i = \{T(v_i) : v_i \text{ and } T(v_i) \text{ are vertexes of } P_p\}.$$

Hence, for any two cycles C_i and C_j, we have

1. $C_i \cap C_j = \emptyset$ or $C_i = C_j$;
2. $\bigcup_i C_i = \{v_1, \ldots, v_p\}$.

If one the vertexes $v_i \in P_p$ is fixed by an elliptic element $T \in \Gamma_p$, this is, if $T(v_i) = v_i$, then all the vertexes of the cycle are fixed by elliptic elements[7] of Γ_p. More precisely, if $T(v_i) = v_i$ and T is elliptic, then $S(v_i) = v_j$ is fixed by the elliptic element STS^{-1}. Thus, its stabilizers G_{v_i} and G_{v_j} of v_i and v_j, respectively, are conjugate, this is, $G_{v_i} = SG_{v_j}S^{-1}$ and, thus, they have the same order (are conjugate subgroups of Γ_p). Therefore, we have the following result:

Theorem 3.1 [2] *Let* P_p *a Dirichlet domain for* Γ_p. *Let* $v_1, \ldots, v_t$ *be the vertexes of one cycle and* $\theta_1, \ldots, \theta_t$ *the internal angles in the respective vertexes. If* m *is the order of the stabilizer in* Γ_p *of some* v_i, *then*

$$\theta_1 + \cdots + \theta_t = \frac{2\pi}{m}. \qquad (2)$$

As a consequence of the Theorem 3.1, we have that if a vertexes of the cycle is not fixed, then point $G_{v_i} = \{Id\}$, so that $m = 1$, $i = 1, \ldots, t$.

The next theorem supplies us a method of determining a set of generators for the group Γ_p.

Theorem 3.2 [4] *Let* $P_p = \mathcal{D}_z(\Gamma_p)$ *be Dirichlet domain for. Let* $\{T_i : T_i \in \Gamma_p\}$ *be the subset of* Γ_p *consisting of elements which pair the edges of* P_p. *Then* $\{T_i : T_i \in \Gamma_p\}$ *is a set of generators for* Γ_p.

We conclued of the Theorems 3.1 and 3.2, that to obtain a discrete group of isomtries Γ_p, from a tessellation $\{p, q\}$, it is necessary to consider a set of pairings of the edges of P_p satisfying to the condition[8] (2). Consequently, it is possible to obtain the genus g of the compact surface that results of those considerations. The genus g is obtained from Euler´s characteristic, this is, $\chi(\mathbb{D}^2/\Gamma_p)$ = number of vertexes − number of edges + number of faces. Thus,

$$\chi(\mathbb{D}^2/\Gamma_p) = \frac{p}{q} - \frac{p}{2} + 1 = 2 - 2g, \qquad (3)$$

for campact surfaces.

We want from Γ_p, to obtain a compact surface $\mathbb{D}^2/\Gamma_p$ with genus g. Firstly, we should have[9], [1],

$$4g \leq p \leq 12g - 6. \qquad (4)$$

Similarly, considering the restriction of the equivalence relation defined in (1) over the set of the p edges of P_p,

$$\{\tau_1, \tau_2, \ldots, \tau_p\},$$

we verify that each equivalence class of edges contains exactly two elements. Therefore, p must necessarily be an even number. Thus, for an edge $\tau_i \in P_p$, there exists unique edges $\tau_j \in P_p$ and an element unique $T \in \Gamma_p$ such that

$$T(\tau_i) = \tau_j \Leftrightarrow T^{-1}(\tau_j) = \tau_i,$$

this is, the equivalence class of τ_i is $\{\tau_i, \tau_j\}$. In this case, we say that T *relates the pair* $\{\tau_i, \tau_j\}$ or that *pairs the edges* τ_i and τ_j. We say too that the edges τ_i and τ_j are *congruent*. Besides, if T relate the pair $\{\tau_i, \tau_j\}$, then T^{-1} too relates it. We will use the symbols

$$T(\tau_i) = \tau_j \Leftrightarrow \tau_i \to \tau_j \Leftrightarrow \{\tau_i, \tau_j\}$$

to indicate that τ_i and τ_j are congruent. Addition in the index of the edge τ_{k+l} is realized-module p.

On the other hand, as each vertex of P_p is covered for q of thoses polygons, we have by Theorem 3.1 that each cycle must have exactly q vertexes[10], so that q must divide p. Those are the conditions that we should use to determine the pairings of the edges of P_p.

[7] In this case, the cycle C_i is called *elliptic cycle*.

[8] If the condition (2) is not satisfied, then we cannot guarantee that the group Γ_p be discrete.

[9] Genus g is obtained from (3).

[10] Vertexes number in a cycle is called *length of the cycle*.

4 Riemann Surface $\mathbb{D}^2/\Gamma_{10\lambda}$

In this section we present the construction of the Fuchsian groups $\Gamma_{10\lambda}$ e, hence, of the Riemann surface $\mathbb{D}^2/\Gamma_{10\lambda}$.

Let P_p be the regular hyperbolic polygon of $p = 10\lambda$ edges associated with the tessellation [11] $\{10\lambda, 2\lambda\}$ and $\lambda \in \{2k : k \in \mathbb{N}\}$ with $k > 1$. We consider

$$\tau_1, \tau_2, \ldots, \tau_{10\lambda} \qquad \text{e} \qquad v_1, v_2, \ldots, v_{10\lambda},$$

the edges and the vertexes of P_p, respectively, disposed in fixed cyclical orders. For each λ, we consider

$$\begin{cases} I_{1,i} = \{(i-1)\lambda + i, \ldots, i(\lambda+1) - 2\}, \\ I_{1,5} = \{4\lambda + 4, \ldots, 5\lambda - 1\}, \end{cases}$$

where $i = 1, \ldots, 4$,and

$$I_1^* = \{\lambda + 1, 2\lambda + 2, 3\lambda + 3, 5\lambda, 6\lambda + 1, 7\lambda + 2, 8\lambda + 3, 10\lambda\},$$

where $I_{1,i}$, $i = 1, 2, 3, 4$, are arithmetic progressions or (AP´s) with $\lambda - 1$ elements and same ratio $r = 1$, while $I_{1,5}$ is an AP with $\lambda - 4$ elements and ratio $r = 1$. For the case $\lambda = 4$, we consider $I_{1,5} = \varnothing$. Let us do then

$$I_1 = \bigcup_{i=1}^{5} I_{1,i}.$$

With those notations, we have:

Theorem 4.1 *Let $\{10\lambda, 2\lambda\}$ be a family of hyperbolic tessellations with regular polygons associated[12] $P_{10\lambda}$, in that $\lambda \in \{2k : k \in \mathbb{N}\}$ with $k > 1$. Then[13]*

$$\tau_i \to \begin{cases} \tau_{5\lambda+i}, & se \quad i \in I_1 \\ \tau_{\lambda+i}, & se \quad i \in I_1^* \end{cases}, \qquad (5)$$

supply pairings for edges of $P_{10\lambda}$, such that $\mathbb{D}^2/\Gamma_{10\lambda}$ results in a compact Riemann surface and orientable of genus $g = \frac{5\lambda-4}{2}$ locally isometric to $\mathbb{D}^2$.

Proof: By (5), we have a total of five cycles. Besides, they are specified for

$$\begin{aligned} C_1 &= \{v_1, v_2, \ldots, v_\lambda, v_{5\lambda+1}, v_{5\lambda+2}, \ldots, v_{6\lambda}\}, \\ C_2 &= \{v_{\lambda+1}, v_{2\lambda+2}, \ldots, v_{4\lambda+4}, v_{5\lambda}, \\ &\quad v_{6\lambda+1}, v_{7\lambda+2}, \ldots, v_{9\lambda+4}, \\ &\quad v_{4\lambda+5}, \ldots, v_{5\lambda-1}, v_{9\lambda+5}, \ldots, v_{10\lambda}\}, \\ C_3 &= \{v_{\lambda+2}, v_{\lambda+3}, \ldots, v_{2\lambda+1}, \\ &\quad v_{6\lambda+2}, v_{6\lambda+3}, \ldots, v_{7\lambda+1}\}, \\ C_4 &= \{v_{2\lambda+3}, v_{2\lambda+4}, \ldots, v_{3\lambda+2}, \\ &\quad v_{7\lambda+3}, v_{7\lambda+4}, \ldots, v_{8\lambda+2}\}, \\ C_5 &= \{v_{3\lambda+4}, v_{3\lambda+5}, \ldots, v_{4\lambda+3}, \\ &\quad v_{8\lambda+4}, v_{8\lambda+5}, \ldots, v_{9\lambda+3}\}. \end{aligned}$$

Cycle C_1 is divided in two blocks. The first block consists of the first vertexes, beginning in the vertex v_1 and ending in the vertex v_λ; the second also contains λ vertexes, that begins in the vertex $v_{5\lambda+1}$ and ending in the vertex $v_{6\lambda}$. The cycles C_3, C_4 and C_5 are organized similarly to the cycle C_1. Already the cycle C_2 is divided in four blocks. The first consists of five vertexes[14], that begins at $v_{\lambda+1}$ and ends at $v_{5\lambda}$, where the indexes of the first four vertexes obey an AP of ratio $r = \lambda+1$. The second contains four vertexes, whose indexes are in AP of ratio $r = \lambda + 1$, beginning at $v_{6\lambda+1}$ and it is going up to $v_{9\lambda+4}$. The third contains $\lambda - 5$ vertexes, where its indexes obey an AP of ratio $r = 1$, beginning at $v_{4\lambda+5}$ and ending at $v_{5\lambda-1}$. Finally, the fourth block contains $\lambda - 4$ vertexes, where your indexes form an AP of ratio $r = 1$, being $v_{9\lambda+5}$ and $v_{10\lambda}$ the first and last vertexes, respectively. Therefore, all the cycles have length 2λ. Hence, by Theorem 3.1, it follows that $\Gamma_{10\lambda}$ is a discrete group of isometries. Thus, by Theorem 2.1 we have that $\mathbb{D}^2/\Gamma_{10\lambda}$ is a compact Riemann surface (locally isometric to $\mathbb{D}^2$)[15] and orientable (g-torus), whose genus by (3) is $g = \frac{5\lambda-4}{2}$. ■

In the Figure 1, we have the polygon P_{60}, associated with the tesselltion $\{60, 12\}$, with the respective pairings of its edges. We obtain of P_{60} the surface $\mathbb{D}^2/\Gamma_{60}$ of genus $g = 13$, locally isometric to $\mathbb{D}^2$.

[11]That means that the internal angles in the respective vertexes of the polygon P_p is $\frac{2\pi}{q} = \frac{\pi}{\lambda}$.

[12]That means that the internal angles in the respective vertexes of the polygon P_p is $\frac{2\pi}{q} = \frac{\pi}{\lambda}$.

[13]The indexes in I_1 will index the generators of $\Gamma_{10\lambda}$ that identify the edges of $P_{10\lambda}$ diametrically opposite. Already the indexes in I_1^* will index the generators that identify the edges that are not diametrically opposite.

[14]For $\lambda = 4$, we order C_2 in the form $C_2 = \{v_5, v_{10}, v_{15}, v_{20}, v_{25}, v_{30}, v_{35}, v_{40}\}$.

[15]According to the pairings in (5), the group $\Gamma_{10\lambda}$ is unprovided parabolic elements and of elliptic elements.

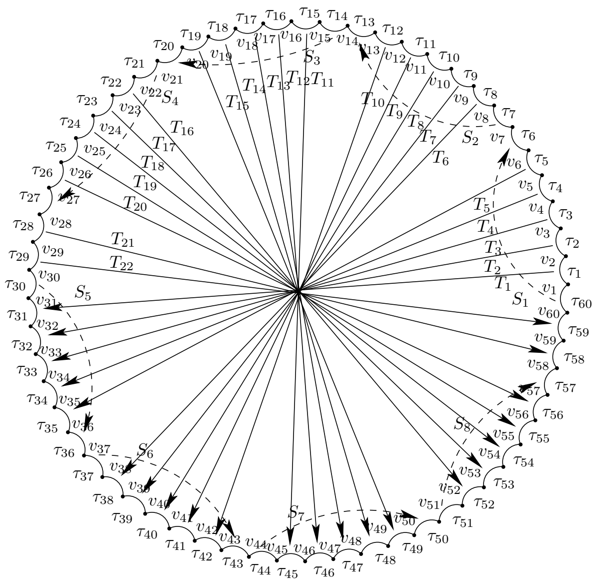

Figure 1: Tessellation $\{10\lambda, 2\lambda\}, \lambda = 6$

Example 4.1 *For $\lambda = 6$, this is, $g = 13$, we have the tessellation $\{60, 12\}$. Thus,*

$$I_{1,1} = \{1,2,3,4,5\}, \quad I_{1,2} = \{8,9,10,11,12\},$$
$$I_{1,3} = \{15,16,17,18,19\}, \quad I_{1,5} = \{28,29\}$$
$$I_{1,4} = \{22,23,24,25,26\},$$

and

$$I_1^* = \{7, 14, 21, 30, 37, 44, 51, 60\}.$$

Therefore, if $i \in I_1 = \bigcup_{i=1}^{5} I_{1,i}$, then the edges pairs are

$$\{\tau_1,\tau_{31}\},\ \{\tau_2,\tau_{32}\},\ \{\tau_3,\tau_{33}\},\ \{\tau_4,\tau_{34}\},$$
$$\{\tau_5,\tau_{35}\},\ \{\tau_8,\tau_{38}\},\{\tau_9,\tau_{39}\},\ \{\tau_{10},\tau_{40}\},$$
$$\{\tau_{11},\tau_{41}\},\ \{\tau_{12},\tau_{42}\},\ \{\tau_{15},\tau_{45}\},\ \{\tau_{16},\tau_{46}\},$$
$$\{\tau_{17},\tau_{47}\},\ \{\tau_{18},\tau_{48}\},\ \{\tau_{19},\tau_{49}\},\ \{\tau_{22},\tau_{52}\},$$
$$\{\tau_{23},\tau_{53}\},\ \{\tau_{24},\tau_{54}\},\{\tau_{25},\tau_{55}\},\ \{\tau_{26},\tau_{56}\},$$
$$\{\tau_{28},\tau_{58}\},\ \{\tau_{29},\tau_{59}\}.$$

If $i \in I_1^*$, then

$$\{\tau_6,\tau_{60}\},\ \{\tau_7,\tau_{13}\},\ \{\tau_{14},\tau_{20}\},\{\tau_{21},\tau_{27}\},$$
$$\{\tau_{30},\tau_{36}\},\{\tau_{37},\tau_{43}\},\ \{\tau_{44},\tau_{50}\},\ \{\tau_{51},\tau_{57}\}.$$

Hence, we obtain the cycles

$$C_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_{31}, v_{32}, v_{33}, v_{34}, v_{35}, v_{36}\},$$
$$C_2 = \{v_7, v_{14}, v_{21}, v_{28}, v_{29}, v_{30}, v_{37}, v_{44}, v_{51}, v_{58}, v_{59}, v_{60}\},$$
$$C_3 = \{v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{30}, v_{39}, v_{40}, v_{41}, v_{42}, v_{43}\},$$
$$C_4 = \{v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{45}, v_{46}, v_{47}, v_{48}, v_{49}, v_{50}\},$$
$$C_5 = \{v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}, v_{52}, v_{53}, v_{54}, v_{55}, v_{56}, v_{57}\}.$$

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