# Interval Valued D-Implications

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## Abstract

The aim of this work is to introduce the concepts of interval D-implication and interval automorphism, analyzing their main properties and establishing the relation between them. Also, interval D-implications are related with punctual D-implications and automorphisms.

## 1 Introduction

Fuzzy set theory [39], is the oldest and most widely reported component of present-day soft computing, which deals with the design of flexible information processing systems [26], with applications in control systems [14], decision making [13], expert systems [36], pattern recognition [26, 3], etc.

On the other hand, Interval Mathematics [27] is a mathematical theory that aims at the representation of uncertain input data and parameters, and the automatic and rigorous controlling of the errors that arise in numerical computations.

The interval-valued fuzzy set theory, which aims at the integration of Fuzzy Theory and Interval Mathematics, has been studied from different viewpoints (see, e.g., [15, 18, 23, 28, 30, 37, 17, 40]). One of these approaches uses membership functions with intervals values, in order to model the uncertainty in the process of determining exact membership grades with the usual fuzzy membership functions. Then, to each element of the universe a closed subinterval of the unit interval is assigned, which approximates the membership degree.

In this work, we adopt the approach introduced in [8, 9], where interval extensions of fuzzy connectives are constructed as their interval representations [35], which considers both correctness (accuracy) and optimality aspects as required in [20]. This approach was also considered in our previous works [7, 5, 6, 32, 33].

In fuzzy set theory, implication functions are usually derived from t-norms and t-conorms in several ways, e.g, S-implications, R-implications, QL-implications and D-implications.[1] The importance of implications is not only because they are used in representing "If ... then" rules in fuzzy systems, but also because their use in performing inferences in approximate reasoning and fuzzy control. This is the main reason for searching many different models to perform this kind of operators.

In particular, D-implications were studied only recently (see, e.g., [24, 25]). The aim of this work is to introduce interval D-implications and to study their relationship with interval automorphisms, and also with punctual D-implications and automorphisms. In Section 2, we review the main concepts related to interval representations. Interval fuzzy t-conorms (t-norms) and negations are presented in Sections 3 and 4, respectively. Interval fuzzy implications and D-implications are discussed in Section 5. Interval automorphisms are presented in Section 6, and their relationship with interval D-implications in Section 7. Section 8 is the Conclusion.

## 2 Interval Representations

Let $\mathbb{U} = \{[a,b] \mid 0 \leq a \leq b \leq 1\}$ be the set of subintervals of $U = [0,1] \subseteq \mathbb{R}$. For $X = [\overline{X}, \underline{X}] \in$

[1] There are also some other methods of generating fuzzy implications (see, e.g, [38]).

$\mathbb{U}$, the projections $l, r : \mathbb{U} \to U$ are defined by

$$l(X) = \overline{X} \text{ and } r(X) = \underline{X}. \quad (1)$$

Among the partial orders that may be defined on $\mathbb{U}$ [12], in this work we consider the following:
- *Product order*: $X \leq Y \Leftrightarrow \underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$.
- *Inclusion order*: $X \subseteq Y \Leftrightarrow \underline{X} \geq \underline{Y}$ and $\overline{X} \leq \overline{Y}$.

**Definition 2.1** $F : \mathbb{U}^n \longrightarrow \mathbb{U}$ *is an* ***interval representation*** *of a function* $f : U^n \longrightarrow U$ *if, for each* $\vec{X} \in \mathbb{U}^n$ *and* $\vec{x} \in \vec{X}$, $f(\vec{x}) \in F(\vec{X})$. [35]

$F : \mathbb{U}^n \longrightarrow \mathbb{U}$ is a *better interval representation* of $f : U^n \longrightarrow U$ than $G : \mathbb{U}^n \longrightarrow \mathbb{U}$, denoted by $G \sqsubseteq F$, if, for each $\vec{X} \in \mathbb{U}^n$, $F(\vec{X}) \subseteq G(\vec{X})$.

**Definition 2.2** [35] *The* ***best interval representation of a real function f****,* $f : U^n \longrightarrow U$*, is the interval function* $\widehat{f} : \mathbb{U}^n \longrightarrow \mathbb{U}$*, defined by*

$$\widehat{f}(\vec{X}) = [\inf\{f(\vec{x}) : \vec{x} \in \vec{X}\}, \sup\{f(\vec{x}) : \vec{x} \in \vec{X}\}] \quad (2)$$

The interval function $\widehat{f}$ is well defined and for any other interval representation $F$ of $f$, $F \sqsubseteq \widehat{f}$. $\widehat{f}$ returns a narrower interval than any other interval representation of $f$. Thus, $\widehat{f}$ has the *optimality property* of interval algorithms [20], when it is seen as an algorithm to compute a real function $f$.

# 3 Interval t-norms and t-conorms

A t-conorm (resp. t-norm) is a function $S : U^2 \to U$ (resp. $T : U^2 \to U$) that is commutative, associative, monotonic and has 0 (resp. 1) as neutral element. In the following, we present the interval generalizations of t-conorms (resp. t-norms), and also two results provided in [8, 10].

**Definition 3.1** *A function* $\mathbb{S} : \mathbb{U}^2 \to \mathbb{U}$ *(resp.* $\mathbb{T} : \mathbb{U}^2 \to \mathbb{U}$*) is an* ***interval t-conorm*** *(resp.* ***t-norm****) if it is commutative, associative, monotonic w.r.t. the product and inclusion order and* $[0, 0]$ *(resp.* $[1, 1]$*) is the neutral element.*

**Proposition 3.2** *If* $S$ *(resp.* $T$*) is a t-conorm (resp. t-norm) then* $\widehat{S} : \mathbb{U}^2 \to \mathbb{U}$ *(resp.* $\widehat{T} : \mathbb{U}^2 \to \mathbb{U}$*) is an interval t-conorm (resp. t-norm). Characterizations of* $\widehat{S}$ *and* $\widehat{T}$ *are given, respectively, by:*

$$\widehat{S}(X, Y) = [S(\underline{X}, \underline{Y}), S(\overline{X}, \overline{Y})], \quad (3)$$
$$\widehat{T}(X, Y) = [T(\underline{X}, \underline{Y}), T(\overline{X}, \overline{Y})]. \quad (4)$$

**Proposition 3.3** *Let* $\mathbb{S}$ *be an interval t-conorm and* $\mathbb{T}$ *an interval t-norm. Then, for all* $X, Y \in \mathbb{U}$*:*

$$\mathbb{S}(X, Y) = [\underline{\mathbb{S}}(\underline{X}, \underline{Y}), \overline{\mathbb{S}}(\overline{X}, \overline{Y})] \quad (5)$$
$$\mathbb{T}(X, Y) = [\underline{\mathbb{T}}(\underline{X}, \underline{Y}), \overline{\mathbb{T}}(\overline{X}, \overline{Y})], \quad (6)$$
$$\underline{\mathbb{S}}(x, y) = l(\mathbb{S}([x, x], [y, y])), \quad (7)$$
$$\overline{\mathbb{S}}(x, y) = r(\mathbb{S}([x, x], [y, y])), \quad (8)$$
$$\underline{\mathbb{T}}(x, y) = l(\mathbb{T}([x, x], [y, y])), \quad (9)$$
$$\overline{\mathbb{T}}(x, y) = r(\mathbb{T}([x, x], [y, y])). \quad (10)$$

*In addition, the functions* $\underline{\mathbb{S}}, \overline{\mathbb{S}}$ *and* $\underline{\mathbb{T}}, \overline{\mathbb{T}}$ *are t-conorms and t-norms, respectively.*

# 4 Interval Fuzzy Negation

A function $N : U \to U$ is a *fuzzy negation* if
**N1** : $N(0) = 1$ and $N(1) = 0$;

**N2** : If $x \geq y$ then $N(x) \leq N(y), \forall x, y \in I$.

Fuzzy negations satisfying the involutive property are called *strong fuzzy negations* [21, 11]:

**N3** : $N(N(x)) = x, \forall x \in U$.

**Definition 4.1** *An interval function* $\mathbb{N} : \mathbb{U} \longrightarrow \mathbb{U}$ *is an* ***interval fuzzy negation*** *if, for any* $X$, $Y$ *in* $\mathbb{U}$*, the following properties hold:*

$\mathbb{N}1$ : $\mathbb{N}([0, 0]) = [1, 1]$ *and* $\mathbb{N}([1, 1]) = [0, 0]$*;*

$\mathbb{N}2$ : *If* $X \geq Y$ *then* $\mathbb{N}(X) \leq \mathbb{N}(Y)$*;*

$\mathbb{N}3$ : *If* $X \subseteq Y$ *then* $\mathbb{N}(X) \supseteq \mathbb{N}(Y)$.

*If* $\mathbb{N}$ *also meets the involutive property, it is said to be a* ***strong interval fuzzy negation****:*

$\mathbb{N}4$ : $\mathbb{N}(\mathbb{N}(X)) = X$, $\forall X \in \mathbb{U}$.

Let $N : U \longrightarrow U$ be a fuzzy negation. A characterization of $\widehat{N}$ is given by:

$$\widehat{N}(X) = [N(\overline{X}), N(\underline{X})]. \quad (11)$$

**Proposition 4.2** $\mathbb{N} : \mathbb{U} \longrightarrow \mathbb{U}$ *is an interval strong fuzzy negation if and only if there exists a strong fuzzy negation* $N$ *such that* $\mathbb{N} = \widehat{N}$ *[2].*

Clearly, in this case, it holds that $N = \underline{\mathbb{N}} = \overline{\mathbb{N}}$.

# 5 Interval D-Implications

Several definitions for fuzzy implication together with related properties have been given (see, e.g., [38, 11, 1, 16, 34]), where a binary function $I : U^2 \longrightarrow U$ is a *fuzzy implication* if, at least, $I$ meets the minimal boundary conditions: $I(1, 1) = I(0, 1) = I(0, 0) = 1$ and $I(1, 0) = 0$. However, other different properties may be required. In this paper, we also consider the following two properties for a fuzzy implications:

**I1a** : If $x \leq z$ then $I(x,y) \geq I(z,y)$;

**I1b** : If $y \leq z$ then $I(x,y) \leq I(x,z)$;

**I2** : $I(1,x) = x$.

Let $S$ be a t-conorm, $N$ be a strong fuzzy negation and $T$ be a t-norm. A Dishkant Implication (D-implication, for short) is a fuzzy implication defined, for all $x, y \in [0,1]$, by [24, 25]:

$$I_{S,T,N}(x,y) = S(T(N(x),N(y)),y). \quad (12)$$

A function $\mathbb{I} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ is an *interval fuzzy implication* if the following conditions hold:

$$\mathbb{I}([1,1],[1,1]) = \mathbb{I}([0,0],[0,0]) = \mathbb{I}([0,0],[1,1]) = [1,1] \text{ and } \mathbb{I}([1,1],[0,0]) = [0,0].$$

The properties **I1a**, **I1b** and **I2** of fuzzy implications can be naturally extended:

$\mathbb{I}1$**a** : If $X \leq Z$ then $\mathbb{I}(X,Y) \geq \mathbb{I}(Z,Y)$;

$\mathbb{I}1$**b** : If $Y \leq Z$ then $\mathbb{I}(X,Y) \leq \mathbb{I}(X,Z)$;

$\mathbb{I}2$ : $\mathbb{I}([1,1],X) = X$.

The proofs of the three following propositions follow directly from the definition of $\widehat{I}$.

From any fuzzy implication, it is possible to obtain canonically an interval fuzzy implication:

**Proposition 5.1** *If $I$ is a fuzzy implication then $\widehat{I}$ is an interval fuzzy implication.*

We can recover the original fuzzy implication from its best interval representation:

**Proposition 5.2** *If $I$ is a fuzzy implication then, for each $x, y \in U$, it holds that $I(x,y) = l(\widehat{I}([x,x],[y,y])) = r(\widehat{I}([x,x],[y,y]))$.*

In the following proposition, the best interval representation of a fuzzy implication is shown to be an inclusion-monotonic function.

**Proposition 5.3** *Let $I$ be a fuzzy implication. For each $X_1, X_2, Y_1, Y_2 \in \mathbb{U}$, if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ then it holds that $\widehat{I}(X_1,Y_1) \subseteq \widehat{I}(X_2,Y_2)$.*

**Theorem 5.4** *If $\mathbb{I} : \mathbb{U}^2 \to \mathbb{U}$ is an inclusion monotonic interval fuzzy implication satisfying $\mathbb{I}1a$ and $\mathbb{I}1b$, then the fuzzy implications $\underline{\mathbb{I}}, \overline{\mathbb{I}} : U^2 \to U$,*

$$\underline{\mathbb{I}}(x,y) = l(\mathbb{I}([x,x],[y,y])) \quad (13)$$
$$\overline{\mathbb{I}}(x,y) = r(\mathbb{I}([x,x],[y,y])), \quad (14)$$

*satisfy the properties* **I1a**, **I1b** *and* $\mathbb{I}(X,Y) = [\underline{\mathbb{I}}(\overline{X},\underline{Y}), \overline{\mathbb{I}}(\underline{X},\overline{Y})]$.

**Proof:** See [4]. ▲

An interval fuzzy implication $\mathbb{I}$ is an **interval D-implication** whenever there are an interval t-conorm $\mathbb{S}$, an interval t-norm $\mathbb{T}$ and a strong interval fuzzy negation $\mathbb{N}$ such that

$$\mathbb{I}(X,Y) = \mathbb{S}(\mathbb{T}(\mathbb{N}(X),\mathbb{N}(Y)),Y)). \quad (15)$$

In this case, we denote $\mathbb{I}$ by $\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}$.

**Theorem 5.5** *Let $S$ be a t-conorm, $T$ be a t-norm and $N$ be a fuzzy negation. If $S$, $T$ and $N$ are continuous then*

$$\mathbb{I}_{\widehat{S},\widehat{T},\widehat{N}} = \widehat{I_{S,T,N}}. \quad (16)$$

**Proof:** Considering $X, Y \in \mathbb{U}$, one has that

$\mathbb{I}_{\widehat{S},\widehat{T},\widehat{N}}(X,Y)$
$= \widehat{S}(\widehat{T}((\widehat{N}(X),\widehat{N}(Y)),Y))$ by Eq.(15)
$= \widehat{S}(\widehat{T}([N(\overline{X}),N(\underline{X})],[N(\overline{Y}),N(\underline{Y})]),Y)$ by Eq.(11)
$= \widehat{S}([T(N(\overline{X}),N(\overline{Y})),T(N(\underline{X}),N(\underline{Y}))],Y)$ by Eq.(4)
$= [S(T(N(\overline{X}),N(\overline{Y})),\underline{Y}),S(T(N(\underline{X}),N(\underline{Y})),\overline{Y})]$ by Eq.(3)

Since, for each $x \in X$ and $y \in Y$, $S(T(N(\overline{X}),N(\overline{Y})),\underline{Y}) \leq S(T(N(x,y)),y) \leq S(T(N(\underline{X}),N(\underline{Y})),\overline{Y})$, then it holds that $\widehat{I_{S,T,N}}(X,Y) \subseteq \mathbb{I}_{\widehat{S},\widehat{T},\widehat{N}}(X,Y)$. On the other hand, if $z \in \widehat{S}(\widehat{T}(N(X),N(Y)),Y)$, then, by the continuity of $S$, there exist $z_1 \in \widehat{T}(N(X),N(Y))$ and $z_2 \in Y$ such that $S(z_1,z_2) = z$. Thus, by the continuity of $N$ and $T$, there exist $z_{1a} \in X$, $z_{1b} \in Y$ such that $T(N(z_{1a}),N(z_{1b})) = z_1$ and, thus, it holds that $S(T(N(z_{1a}),N(z_{1b})),z_2) = z$. If $z_2 \leq z_{1a}$, then one has that $S(T(N(z_2),N(z_{1b})),z_2),) \geq z$, and by the commutativity of $T$, it holds that $S(T(N(z_{1b}),N(z_2)),z_2),) \geq z$. In addition, if $z_{1b} \leq z_2$, then it is valid that $S(T(N(z_{1a}),N(z_{1b})),z_{1b}),) \leq z$. So, one has that $z \in [I_{S,T,N}(z_{1a},z_{1b}),I_{S,T,N}(z_{1b},z_2)] \subseteq \{I_{S,N,T}(x,y) : x \in X, y \in Y\}$, and $\{I_{S,T,N}(x,y) : x \in X, y \in Y\} \subseteq \widehat{I_{S,T,N}}(X,Y)$. Therefore, it follows that $\mathbb{I}_{\widehat{S},\widehat{T},\widehat{N}}(X,Y) = \widehat{S}(\widehat{T}(\widehat{N}(X),\widehat{N}(Y)),Y) \subseteq \widehat{I_{S,T,N}}(X,Y)$. ▲

**Corollary 5.6** *If $I$ is a continuous D-Implication then $\widehat{I}$ is an interval D-implication.*

**Proof:** It is straightforward, following from Theorem 5.5. ▲

**Proposition 5.7** *If $\mathbb{I}$ is an interval D-implication then the properties $\mathbb{I}1a$ and $\mathbb{I}2$ hold.*

**Proof:** Consider $\mathbb{I}$ as an interval D-implication and $X, Y, Z \in \mathbb{U}$.

$\mathbb{I}1a$: Based on the monotonicity of the interval t-norm $\mathbb{T}$, the interval t-conorm $\mathbb{S}$ and interval negation $\mathbb{N}$, if $X \leq Z$ then $\mathbb{S}(\mathbb{T}(\mathbb{N}(X), \mathbb{N}(Y)), Y) \geq \mathbb{S}(\mathbb{T}(\mathbb{N}(Z), \mathbb{N}(Y)), Y)$. Therefore, it holds that $\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}(X, Y) \geq \mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}(Z, Y)$.

$\mathbb{I}2$: One has that $\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}([1,1], Y) = \mathbb{S}(\mathbb{T}(\mathbb{N}([1,1]), \mathbb{N}(Y)), Y) = \mathbb{S}(\mathbb{T}([0,0], \mathbb{N}(Y)), Y)$. By the monotonicity of $\mathbb{T}$, it holds that $\mathbb{S}(\mathbb{T}([0,0], \mathbb{N}(Y)), Y) = \mathbb{S}([0,0], Y) = Y$. Then, it follows that $\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}([1,1], Y) = Y$. ▲

Denote by $\mathcal{C}(S)$, $\mathcal{C}(T)$, $\mathcal{C}(N)$ and $\mathcal{C}(I)$ the classes of continuous t-conorms, continuous t-norms, strong fuzzy negations and continuous D-implications, respectively. The related interval extensions are indicated by $\mathcal{C}(\mathbb{S})$, $\mathcal{C}(\mathbb{T})$, $\mathcal{C}(\mathbb{N})$ and $\mathcal{C}(\mathbb{I})$, respectively. The results presented in sections 3 and 5, together with Theorem 5.5, state the commutativity of the diagram in Fig. 1.

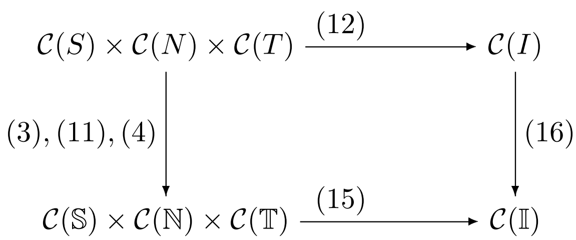


Figure 1: Classes of interval D-implications

**Theorem 5.8** *If $\mathbb{I}$ is an interval fuzzy D-implication, then the functions $\underline{\mathbb{I}}, \overline{\mathbb{I}} : U^2 \rightarrow U$, defined by $\underline{\mathbb{I}}(x, y)) = l(\mathbb{I}([x, x], [y, y]))$ and $\overline{\mathbb{I}}(x, y)) = r(\mathbb{I}([x, x], [y, y]))$, are D-implications.*

**Proof:** We prove the first case. The second one is analogous. Considering $x, y \in U$, one has that

$$
\begin{aligned}
&\underline{\mathbb{I}}(x, y) \\
&= l(\mathbb{I}([x, x], [y, y])) \\
&= l(\mathbb{S}(\mathbb{T}(\mathbb{N}([x, x]), \mathbb{N}([y, y])), [y, y])) \text{ by hypothesis} \\
&= l(\mathbb{S}(\mathbb{T}([N(x), N(x)], [N(y), N(y)]), [y, y])) \\
&\qquad \text{by Eq.(11) and Prop. 4.2} \\
&= l[\underline{\mathbb{S}}(\underline{\mathbb{T}}(N(x), N(y)), y), \overline{\mathbb{S}}(\overline{\mathbb{T}}(N(x), N(y)), y)] \\
&\qquad \text{by Eq.(9,10)} \\
&= \underline{\mathbb{S}}(\underline{\mathbb{T}}(N(x), N(y)), y) \text{ by Eq.(1)} \\
&= I_{\underline{\mathbb{S}},\underline{\mathbb{T}},N}(x, y) \text{ by Eq.(12)}
\end{aligned}
$$

▲

The reconstruction of an interval fuzzy D-implication (i.e., the converse of Theorem 5.8) is not possible considering just the fuzzy implications $\underline{\mathbb{I}}$ and $\overline{\mathbb{I}}$. We introduce the following proposition in order to get the condition for the reconstruction.

**Proposition 5.9** *Let $\mathbb{I}$ be an interval D-implication satisfying the property $\mathbb{I}1b$. Then it holds that $\mathbb{I}(X, Y) = [\underline{\mathbb{I}}(\overline{X}, \underline{Y}), \overline{\mathbb{I}}(\underline{X}, \overline{Y})]$.*

**Proof:** Since $\mathbb{S}$, $\mathbb{T}$ and $\mathbb{N}$ are inclusion monotonic, $\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}$ also is inclusion monotonic. Thus, because $\mathbb{I}$ satisfies the property $\mathbb{I}1b$, then from Prop. 5.7 and Theorem 5.4, it follows that $\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}(X, Y) = [\underline{\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}}(\overline{X}, \underline{Y}), \overline{\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}}(\underline{X}, \overline{Y})]$. ▲

# 6 Interval Automorphism

**Definition 6.1** *A mapping $\rho : U \longrightarrow U$ is an **automorphism** if it is bijective and monotonic, that is, $x \leq y \ \Rightarrow \ \rho(x) \leq \rho(y)$* [22, 29].

The inverse of an automorphism is also an automorphism. Based on [21, 11], the **action of an automorphism $\rho$ on a function** $f : U^n \rightarrow U$, denoted by $f^\rho$, is defined as

$$f^\rho(x_1, \ldots, x_n) = \rho^{-1}(f(\rho(x_1), \ldots, \rho(x_n))). \quad (17)$$

As it is well known (see, e.g., [31, 11]), the action of $\rho$ preserves the fuzzy connectives, that is, $S^\rho$,$T^\rho$, $N^\rho$ and $I^\rho$ are fuzzy t-conorm, t-norm, (strong) negation and implication, respectively.

**Proposition 6.2** *Let $S$ be a t-conorm, $T$ be a t-norm and $N$ be a strong negation. Then it holds that $I_{S^\rho,T^\rho,N^\rho} = I^\rho_{S,T,N}$.*

**Proof:** Considering $x, y \in U$, one has that

$$
\begin{aligned}
&I_{S^\rho,T^\rho,N^\rho} \\
&= S^\rho(T^\rho(N^\rho(x), N^\rho(y)), y) \text{ by Eq.(12)} \\
&= S^\rho(T^\rho(\rho^{-1}(N(\rho(x)))), \rho^{-1}(N(\rho(y))), y) \\
&\qquad \text{by Eq.(17)} \\
&= S^\rho(\rho^{-1}(T(\rho(\rho^{-1}(N(\rho(x)))), \\
&\quad \rho(\rho^{-1}(N(\rho(y)))))), y) \text{ by Eq.(17), Def. 6.1} \\
&= \rho^{-1}(S(\rho(\rho^{-1}(T(N(\rho(x)), N(\rho(y))))), \rho(y)) \\
&\qquad \text{by Eq.(17)} \\
&= \rho^{-1}(S((T(N(\rho(x)), N(\rho(y))))), \rho(y)) \text{ by Def. 6.1} \\
&= \rho^{-1}(I_{S,T,N}(\rho(x), \rho(y))) \text{ by Eq.(12)} \\
&= I^\rho_{S,T,N}(x, y) \text{ by Def. 17}
\end{aligned}
$$

▲

**Corollary 6.3** *If $I$ is a D-implication then $I^\rho$ is also a D-implication.*

**Proof:** It follows from the definition of D-implication and Prop. 6.2. ▲

A mapping $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is an **interval automorphism** if it is bijective and monotonic w.r.t. the product order, i.e., $X \leq Y \Rightarrow \varrho(X) \leq \varrho(Y)$ [18, 19].

**Theorem 6.4** *$\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is an interval automorphism if and only if there is an automorphism $\rho : U \to U$, such that $\varrho = \widehat{\rho}$ and*

$$\widehat{\rho}(X) = [\rho(\underline{X}), \rho(\overline{X})]. \quad (18)$$

**Proof:** See [18, Theorem 2], [8, Theorem 5.2]. ▲

Thus, each interval automorphism is the best interval representation of some automorphism.

The **action of an interval automorphism $\varrho$ on an interval function** $F : \mathbb{U}^n \to \mathbb{U}$ is defined as

$$\begin{aligned} & F^{\varrho}(X_1, X_2, \ldots, X_n) \quad (19) \\ & = \varrho^{-1}(F(\varrho(X_1), \varrho(X_2), \ldots, \varrho(X_n))). \end{aligned}$$

The action of $\varrho$ preserves the interval fuzzy connectives, that is, $\mathbb{S}^{\varrho}$,$\mathbb{T}^{\varrho}$, $\mathbb{N}^{\varrho}$ and $\mathbb{I}^{\varrho}$ are also interval fuzzy t-conorm, t-norm, (strong) negation and implication, respectively [8, 10, 4].

**Lemma 6.5** *Let $\rho$ be an automorphism. Then it holds that*

$$\widehat{\rho^{-1}} = \widehat{\rho}^{-1}. \quad (20)$$

**Proof:** Its sufficient to prove that $\widehat{\rho} \circ \widehat{\rho^{-1}} = Id$ ($\widehat{\rho^{-1}} \circ \widehat{\rho} = Id$ follows from the fact that $\rho^{-1}$ is also an automorphism and $(\rho^{-1})^{-1} = \rho$).

$$\begin{aligned} & \widehat{\rho} \circ \widehat{\rho^{-1}}(X) \\ & = \widehat{\rho}([\rho^{-1}(\underline{X}), \rho^{-1}(\overline{X})]) \text{ Eq.(18)} \\ & = [\rho \circ \rho^{-1}(\underline{X}), \rho \circ \rho^{-1}(\overline{X})] \text{ Eq.(18)} \\ & = X \text{ by the bijectivity of } \rho \end{aligned}$$

▲

**Theorem 6.6** *Let $I$ be a fuzzy implication and $\rho$ be an automorphism. Then it holds that $\widehat{I^{\rho}} = \widehat{I}^{\widehat{\rho}}$.*

**Proof:** Considering $X, Y \in \mathbb{U}$, one has that

$$\begin{aligned} & \widehat{I^{\rho}}(X, Y) \\ & = [\inf\{I^{\rho}(x, y) \mid x \in X \wedge y \in Y\}, \\ & \quad \sup\{I^{\rho}(x, y) : x \in X \wedge y \in Y\}] \text{ by Eq.(2)} \\ & = [\inf\{\rho^{-1}(I(\rho(x), \rho(y))) \mid x \in X \wedge y \in Y\}, \\ & \quad \sup\{\rho^{-1}(I(\rho(x), \rho(y))) \mid x \in X \wedge y \in Y\}] \text{by Eq.(17)} \\ & = [\rho^{-1}(\inf\{I(\rho(x), \rho(y)) \mid x \in X \wedge y \in Y\}), \\ & \quad \rho^{-1}(\sup\{I(\rho(x), \rho(y)) \mid x \in X \wedge y \in Y\})] \\ & \qquad \text{by continuity of } \rho^{-1} \\ & = \widehat{\rho^{-1}}([\inf\{I(\rho(x), \rho(y)) \mid x \in X \wedge y \in Y\}, \\ & \quad \sup\{I(\rho(x), \rho(y)) \mid x \in X \wedge y \in Y\}]) \text{by Eq. ( 17)} \\ & = \widehat{\rho^{-1}}([\inf\{I(x, y) \mid x \in \rho(X) \wedge y \in \rho(Y)\}, \\ & \quad \sup\{I(x, y) \mid x \in \rho(X) \wedge y \in \rho(Y)\}]) \\ & \qquad \text{by the continuity of } \rho \\ & = \widehat{\rho^{-1}}([\inf\{I(x, y) \mid x \in \widehat{\rho}(X) \wedge y \in \widehat{\rho}(Y)\}, \\ & \quad \sup\{I(x, y) \mid x \in \widehat{\rho}(X) \wedge y \in \widehat{\rho}(Y)\}]) \\ & \qquad \text{by the continuity of } \rho \\ & = \widehat{\rho^{-1}}(\widehat{I}(\widehat{\rho}(X), \widehat{\rho}(Y))) \text{ by Eq.(2)} \\ & = \widehat{\rho}^{-1}(\widehat{I}(\widehat{\rho}(X), \widehat{\rho}(Y))) \text{by Eq.(20)} \\ & = \widehat{I}^{\widehat{\rho}}(X, Y) \text{by Eq.(19)} \end{aligned}$$

▲

In the following, we apply the results concerned with the generation of new interval D-implications based on the action of interval automorphisms.

# 7 Relation between Interval Automorphism and Interval D-implication

In the following, we prove that the action of an interval automorphism preserves interval D-implications.

**Theorem 7.1** *Let $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ be an interval automorphism and $\mathbb{I} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ be an interval D-implication. Then $\mathbb{I}^{\varrho} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ is also an interval D-implication.*

**Proof:** It is sufficient to prove that the following equation holds:

$$\mathbb{I}^{\varrho}_{\mathbb{S},\mathbb{T},\mathbb{N}}(X, Y) = \mathbb{S}^{\varrho}(\mathbb{T}^{\varrho}(\mathbb{N}^{\varrho}(X), \mathbb{N}^{\varrho}(Y)), Y). \quad (21)$$

Considering $X, Y \in \mathbb{U}$, we have that

$$\begin{aligned} & \mathbb{I}^{\varrho}_{\mathbb{S},\mathbb{T},\mathbb{N}}(X, Y) \\ & = \varrho^{-1}(\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}(\varrho(X), \varrho(Y))) \text{ by Eq.(19)} \\ & = \varrho^{-1}(\mathbb{S}(\mathbb{T}(\mathbb{N}(\varrho(X)), \mathbb{N}(\varrho(Y))), \varrho(Y)) \text{ by Eq.(15)} \\ & = \varrho^{-1}(\mathbb{S}(\mathbb{T}(\varrho(\varrho^{-1}(\mathbb{N}(\varrho(X)))), \varrho(\varrho^{-1}(\mathbb{N}(\varrho(Y))))), \varrho(Y)) \\ & \qquad \text{by the bijectivity of } \varrho \\ & = \varrho^{-1}(\mathbb{S}(\varrho(\varrho^{-1}(\mathbb{T}(\varrho(\mathbb{N}^{\varrho}(X)), \varrho(\mathbb{N}^{\varrho}(Y)))))), \varrho(Y)) \\ & \qquad \text{by Eq.(19)} \\ & = \varrho^{-1}(\mathbb{S}(\varrho(\mathbb{T}^{\varrho}(\mathbb{N}^{\varrho}(X), \varrho\mathbb{N}^{\varrho}(Y)), \varrho(Y))) \text{ by Eq.(19)} \\ & = \mathbb{S}^{\varrho}(\mathbb{T}^{\varrho}(\mathbb{N}^{\varrho}(X), \mathbb{N}^{\varrho}(Y)), Y) \text{ by Eq.(19)} \end{aligned}$$

▲

According to Theorem 6.6, the commutative diagram pictured in Fig. 2 holds.

Based on Theorems 6.6 and 7.1, (interval) D-implications and (interval) automorphisms can be seen as objects and morphisms of the category $\mathfrak{C}(\mathcal{C}(I), Aut(I))$ ($\mathfrak{C}(\mathcal{C}(\mathbb{I}), Aut(\mathbb{I}))$), respectively. In a categorical approach, the action of an interval automorphism on an interval D-implication can be conceived as a covariant

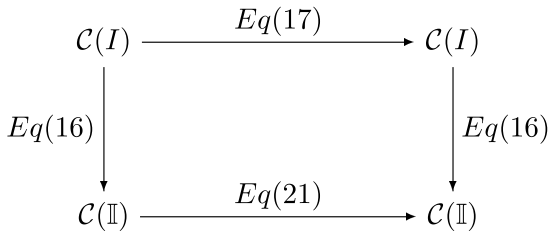


Figure 2: Commutative classes of interval D-implications and automorphisms

functor whose application over the D-implications and automorphisms in $\mathfrak{C}(\mathcal{C}(I), Aut(I))$ returns the related best interval representation in $\mathfrak{C}(\mathcal{C}(\mathbb{I}), Aut(\mathbb{I}))$. Therefore, interval automorphisms can be used to deal with optimality of interval fuzzy algorithms.

## 8 Conclusion and Final Remarks

Considering the importance the study of fuzzy implications for performing inferences in approximate reasoning and fuzzy control, the results presented in this paper contribute firstly for the general study on D-implications, whose main features appeared only recently as subject of some important works (see, e.g., [24, 25]).

However, our main interest was on the definition of interval-valued D-implications, continuing our previous works [7, 5, 6, 32, 33] on the study of the various interval-valued implication functions that are derived form interval t-norms and interval t-conorms. In this context, we proved many important properties of interval D-implications.

Finally, the paper establishes the relationship between (interval) D-implications and (interval) automorphisms, showing that the action of (interval) automorphisms preserve (interval) D-implications.

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