Topological Projects of Modulations on Surfaces

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Abstract: *Here are presented procedures for constructing topological projects of modulations (on bi-dimensional oriented compact variety) through minimal embedding of graphs asso ciated to DMC, particulary of the bipartite complete graphs $K_{s,s}$, $s = 2, 4, \cdots, 7$, which make it possible to describe accurately the type of decomposition caused on the surface by the embedding action.*

1 Introduction

The problem treated in this paper is part of an integrate system of digital signal transmission proposed by [3], in which the coding and modulation blocs are projected from a discrete memory-less channel (DMC) model, aiming at that these components act compatibly, eliminating the additional devices which are usually utilized for correcting distortions introduced in the system caused by the random choice of these components.

For one side, the space for modulation projects is taken in the universe of surfaces utilizing the graph embedding theory. For the other, the coding system is projected from the homology group of the surface on which the DMC is embedded. Despite of both keep relationship with the theory of the surfaces, they have distinct mathematical aspects: modulation is a problem related to graph theory, thus belonging to the Graph Theory [3], while coding belongs to the Algebraic Topology environment. The present paper deals with the modulation issue only.

The main idea is to obtain a geometric model on a surface Ω, i.e., a spatial model originated through 2-cel embedding of the graph associated to the channel, as a basic element for the project of a modulation system. Obtained this model, it is easy to construct the dual, also a 2-cell embedding on Ω, the place upon which, in fact, the modulation project is constructed, where the constellation signals would be in the center of each area of the dual, which would be the decision regions (Voronoy's regions) of each signal.

In this paper, it will be constructed particularly models for modulation projects based on oriented embedding of bipartite complete graphs $K_{s,s}$, $s = 2, 3, \cdots, 7$, which are the graphs associated to the most common models of DMC channels.

2 Concepts and Definition

Polygon representations of a surface are commonly referred as plane models of the surface. Seifert-Therlfall [4], utilizing combinatorial topology, showed that every compact surface is the quotient space of a polygon for an equivalence relationship according to which the edges that constitute the polygon boundary are identified two by two. The polygon is a $2n$-gon and, in the case of orientable compact surfaces, each surface class can be represented by a $2n$-gon indicated by a word referred to as normal form. If S and mT are respectively the sphere and a surface homeomorphic to the connected sum of m surfaces equaling the torus, their respective words in the normal form are $S \equiv aa^{-1}$ and $mT \equiv a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_m b_m a_m^{-1} b_m^{-1}$, whose orientations are indicated in Figura 1.

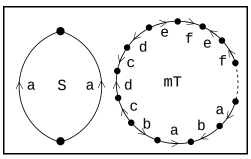

Figura 1: Normal forms of S and mT

A *complete bipartite graph* $K_{s,s}$ is a graph formed by two sets of vertexes, U =

$\{u_1, \cdots, u_s\}$ and $V = \{v_1, \cdots, v_s\}$, with each vertex of U connected to all vertexes of V. The minimum oriented genus of $K_{s,s}$ is given by $\gamma = \{(s-2)^2/4\}$, where the braces denote greatest integer function [7].

3 Embedding on Plane and Space Models

An embedding of graph on surface may be accomplished in the plane model or in the space model [6]. To describe the topological embedding, it is easier in the plane model, however, when this is the normal form the minimal embedding of mT, $m \geq 2$, is of an odd difficulty.

This sort of construction becomes possible when the orientation of the polygon is a result of cutting the homology curves of the surfaces, but the resulting word is in a natural form, which is represented by $4m$-gon.

Definition **1** *The* natural form *of a* $4m$*-gon is any polygon oriented according to the word*

$$a_1 b_1 a_1^{-1} a_2^{-1} \cdots b_m^{-1} a_m b_m b_{m-1}^{-1} a_{m-1} \cdots a_2 b_2 b_1^{-1} \quad (1)$$

which is, in fact, a word associated to a surface homeomorphic to mT, as follows.

Proposition **1** *Let* Ω *be a surface associated to an oriented polygon represented by the word of equality (1). Then* Ω *is homeomorphic to* mT.

Proof: If $m = 1$, nothing to be demonstrated. If $m = 2$, applying the defined operations in [1], it results

$$\begin{aligned}\omega(\Omega) &\equiv a_1 b_1 a_1^{-1} a_2^{-1} b_2^{-1} a_2 b_2 b_1^{-1} \\ &\equiv b_1 a_1^{-1} a_2^{-1} b_2^{-1} a_2 b_2 b_1^{-1} a_1 \\ &\equiv x_1 b_1 x_1^{-1} b_1^{-1} a_2^{-1} b_2^{-1} a_2 b_2 \equiv 2T.\end{aligned}$$

The general case is obtained similarly [3]. ■

Since the normal and natural forms of both the sphere and torus are identical, the difficulties of obtaining a graph embedding on one or another type of model are equivalent. In Figure 2, identical minimum oriented embeddings obtained without any specific method are illustrated.

The problem presented at the beginning of this paper was to obtain oriented minimal embeddings of $K_{s,s}$ and identify the types of regions of such embeddings. A graph embedding on Ω is identified through either its rotation

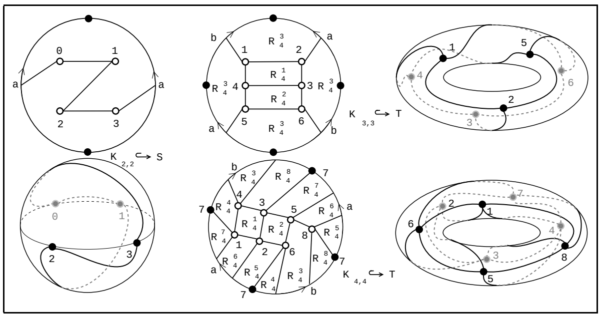

Figura 2: Plane and space models of minimum embedding of $K_{s,s}$ on S and T

system or the paths that define the boundaries (or orbits) of the graph regions. Since the orbit concept is more related to modulation it will be the identifier of the embedding.

Two embeddings of a graph (one in the plane model and the other in the space model) are said *identical* when there exists a one-to-one correspondence between their regions such that the oriented boundaries of the corresponding regions are identical as oriented paths of the graph.

In Figure 2, the plane and space models present identical rotation systems and orbits, as is shown in Table 1.

Ω	Rotate Systems	Orbits
S	$1\{24\}$, $2\{13\}$	$R_4^1 = 1234$, $R_4^2 = 1234$
	$3\{24\}$, $4\{13\}$	
	$1\{246\}$, $2\{153\}$	$R_4^1 = 1234$, $R_4^2 = 3654$
T	$3\{264\}$, $4\{135\}$	$R_6^3 = 214563$
	$5\{246\}$, $6\{153\}$	$R_4^4 = 1256$
	$1\{2684\}$, $2\{1357\}$	$R_4^1 = 1432$, $R_4^2 = 3456$
T	$3\{2468\}$, $4\{1753\}$	$R_4^3 = 5816$, $R_4^4 = 1276$
	$5\{2864\}$, $6\{1735\}$	$R_4^5 = 3678$, $R_4^6 = 2345$
	$7\{2486\}$, $8\{1537\}$	$R_4^7 = 2547$, $R_4^8 = 1874$

Tabela 1: Rotate Systems and Embedding Region Orbits of the Figure 2

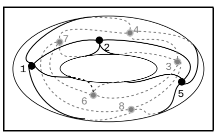

Figura 3: Embedding $K_{4,4} \hookrightarrow T$

To outline a graph embedding on a surface is not a simple task. Relatively small graphs can result in different embeddings. For instance, the rotation systems of $K_{4,4} \hookrightarrow T$ in Figures 2 and 3 are different. The last one has a rotation system

$$1\{2684\},2\{1537\},3\{2684\},4\{1735\},$$
$$5\{1486\},6\{1735\},7\{2486\},8\{1537\},$$

therefore different of the rotation system of $K_{4,4} \hookrightarrow T$ in Table 1.

If the embedding is on the plane model of the mT surface, $m \geq 2$, the identification of the orbits is less complex, yet not simple. However, if the space model of the embedding is available it can be utilized for constructing an identical embedding on the plane model of mT.

Obtained $\gamma(K_{s,s})$, $K_{s,s} \hookrightarrow \gamma T$ will accomplished through the following procedures:

1) Utilize Method 1 for constructing Γ_e, which is the embedding of $K_{s,s}$ on the space model of γT;

2) Utilize Method 2 for constructing Γ_p, which is the embedding of $K_{s,s}$ on the plane model of γT, consequently identical to Γ_e;

3) Utilize Method 3 for mapping the regions (or orbits).

Method **1** *Let* $\{u_1 \cdots, u_s, v_1, \cdots, v_s\}$ *be the vertexes of* $K_{s,s}$. *If* s *is even,* Γ_e *is obtained though the following:*

1) Split the space model of γT *in two surfaces with boundaries identical to* $\frac{\gamma}{2}T_1$ *(models (a) and (b) in Figure 4);*

2) On $\frac{\gamma}{2}T_1$ *distribute* $u_1, \cdots, u_s$ *near the boundary and* $v_1, \cdots, v_{s/2}$ *afar from the boundary, according to embedding (a) in Figure 4;*

3) Connect each vertex $v_1, \cdots, v_{s/2}$ *through a curve on* $\frac{\gamma}{2}T_1$ *to all the vertexes* $u_1, \cdots, u_s$, *so that* $K_{s,s/2} \hookrightarrow \frac{\gamma}{2}T_1$ *is obtained (embedding (a) in Figure 4);*

4) Reproduce $K_{s,s/2} \hookrightarrow \frac{\gamma}{2}T_1$ *as mirror image then unite both images to obtain* Γ_e, *as shown (c) in Figure 4.*

The distributions of the vertexes $u_i's$ and $v_i's$ on $4T$ in the Figura 4 are aimed at to construct the embedding in two identical halves of $4T$.

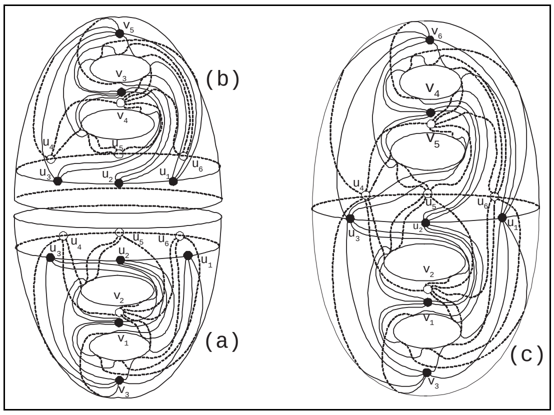

Figura 4: Space model embedding of $K_{6,6} \hookrightarrow 4T$

Obtained one of them, it is possible to copy it on the other half and unite them to obtain $K_{6,6} \hookrightarrow 4T$.

If s is odd, distribute $v_1, \cdots, v_{(s+1)/2}$ as in step 1) of Method 1, them follow 2) and 3), eliminating, in the second half of the graph, one vertex v_i together with al its connections, thus obtaining Γ_e.

Once the homology curves of γT correspond to the edges of 4γ-gon, a cut on this curve gives the plane model of γT oriented in the natural form (1). This allows to obtain Γ_p from Γ_e, as described in Method 2.

Method **2** Γ_p *is obtained through the following steps:*

1) Construct on Γ_e *the homology curves;*

2) Duplicate the homology curves;

3) Label orderly the intersections of $K_{s,s}$ *with the edges of* ω*;*

4) Transfer these labels to the oriented 4γ*-gon;*

5) Distribute the vertexes of $K_{s,s}$ *inside the* 4γ*-gon and utilize the labels in 4) to obtain* Γ_p. *This means to construct the connections of* $K_{s,s}$ *obeying the rotation system of* Γ_e *passing by the boundaries of* 4γ*-gon according to the labels in step 3) (Figure 5).*

Some paths do not cross the edges of the 4γ-gon, that is, they have either direct connections, as in the case of edge (u_4, v_3), or can cross one or more edges of the 4γ-gon, as it occurs, for instance, with (u_1, v_1) which crosses the 4γ-gon boundary at b_2, b_1 and a_1 in the

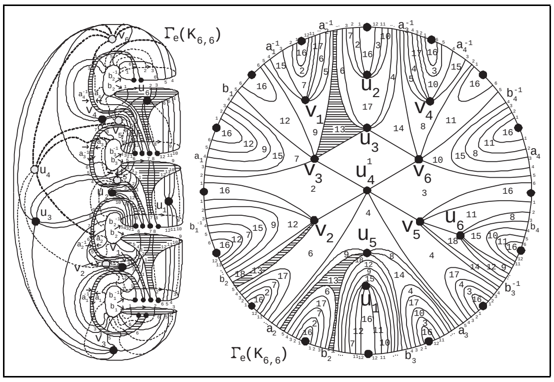

Figura 5: Plane model embedding of $K_{6,6} \hookrightarrow 4T$

interceptions labelled by 10, 4 and 3, respectively.

Method **3** *Identify the orbits of* Γ_p *through the following: choose one edge of* $K_{s,s} \hookrightarrow \gamma T$. *Then follow the orbit clockwisely until reach the end vertex of the next edge; if, for instance, one edge of a mapped orbit reaches the boundary* a_i *of the* 4γ*-gon, it will be automatically connected to the boundary* a_i^{-1} *by a labelled intersection (see step 2) in Method 2), repeat the procedure until reach the first vertex of the initial edge.*

The complete mapping of an orbit that defines i-region R_k^i (i = orbit label; k = number of the edges of the orbits) is done by

$$R_k^i = c_1\Lambda_1 d_2\Lambda_2 c_3\Lambda_3 d_4 \cdots c_{k-1}\Lambda_{k-1} d_k\Lambda_k c_1,$$

$$c_t \in U,\ \forall t \in \{1,3,5,\cdots,k-1\},$$
$$d_r \in v,\ \forall r \in \{2,4,6,\cdots,k\},$$
$$\text{with } k \text{ even } \geq 4;$$

$$\Lambda_j = \left[\frac{\eta_1^{\rho_1}\eta_1^{\rho_1^{-1}}}{t_1^j}\,\frac{\eta_2^{\rho_2}\eta_2^{\rho_2^{-1}}}{t_2^j}\cdots\frac{\eta_q^{\rho_q}\eta_q^{\rho_{1q}^{-1}}}{t_q^j}\right],$$

$$\Lambda_j = \emptyset,\ \text{if } (c_j, d_{j+1}) \cap \omega = \emptyset = (d_j, c_{j+1}) \cap \omega;$$
t_i^j = label of the corresponding intersection (see step 2 in Method 2);
$$i \in \{1,2,\cdots,q\} \text{ and } j \in \{1,2,3,\cdots,k\};$$
$$\eta_h \in \left\{a_1, a_1^{-1}, \cdots a_\gamma, a_\gamma^{-1}, b_1, b_1^{-1}, \cdots b_\gamma, b_\gamma^{-1}\right\};$$
$$\rho_h \in \{-1,1\},\ \forall h \in \{1,2,\cdots,q\}.$$

From Figure 5, the 13-th mapped orbit of $K_{6,6} \hookrightarrow 4T$ is done by

$$R_4^{13} = u_3 v_3 \left[\frac{a_2^{-1}a_2}{6}\right] u_3 \left[\frac{b_2^{-1}b_2}{6}\right] v_2 \left[\frac{b_2 b_2^{-1}}{5}\,\frac{a_2 a_2^{-1}}{5}\right] u_3,$$

or a simplified manner, $R_4^{13} = (u_3, v_3, u_5, v_2)$.

After the complete mapping of $K_{s,s} \hookrightarrow 4T$ (Figure 5), it was verified that the embedding resulted in a geometrically uniform signal constellation $18R_4$.

Space and plane models of minimum embeddings of the graphs $K_{5,5}$ and $K_{7,7}$ are illustrated in Figures 6 and 7, where signal constellations are both of type geometrically non uniforms given by

$$K_{5,5} \hookrightarrow 3T \equiv 10R_4 + R_8 \text{ and}$$
$$K_{7,7} \hookrightarrow 7T \equiv 21R_4 + R_6 + R_8.$$

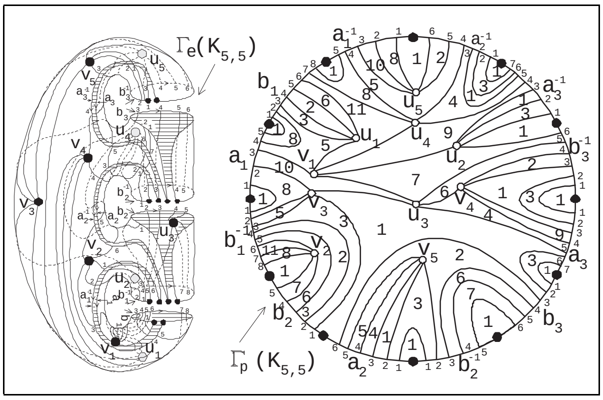

Figura 6: Plane and space models of $K_{5,5} \hookrightarrow 3T$ minimal embedding

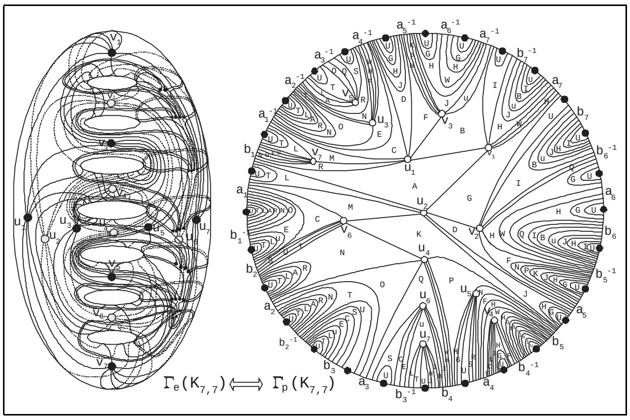

Figura 7: Plane and space models of $K_{7,7} \hookrightarrow 7T$ minimal embedding

4 Acknowledgments

The authors would like to express their thank to Odaci F. de Oliveira for review in the text and their valuable suggestions which have contributed to construct the orbit mapping algorithm.

5 Conclusion

The procedures presented in this paper allow to construct projects of modulations for constellations of s-signals, $s = 2,3,\cdots,7$, originated from minimal embeddings of graph associated to the DMC. They are projects constructed on space (Method 1) and plane (Method 2) models of one oriented surface. The complete mapping of orbits in Method 3, identify the type of embedding composition, and bears all necessary information to construct the corresponding embedding on plane and space model of surfaces and, can be used either in cryptography or in error-correcting code systems. The examples in Figures. 5, 6 and 7 are even more difficult to construct than the embeddings of channels actually in use; this means that is possible to project all of them through these procedures.

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