

On a Nonlocal Elliptic System of p-Kirchhoff-Type Under Neumann Boundary Condition

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Abstract: *In this paper we investigate questions of existence of solution for the system*

$$\begin{cases} - \left[M_1 \left(\int_{\Omega} |\nabla u|^p \right) \right]^{p-1} \Delta_p u = f(u, v) + \rho_1(x), & \Omega \\ - \left[M_2 \left(\int_{\Omega} |\nabla v|^p \right) \right]^{p-1} \Delta_p v = g(u, v) + \rho_2(x), & \Omega \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & \partial\Omega. \end{cases}$$

Motivated by a problem in Costa [3], who studies a single local equation, we study the above problem by using variational methods. Since we will work in the space $W^{1,p}(\Omega)$ the functional associated to the above problem will not be coercive. So, we have to consider the Poincaré-Wirtinger's inequality in the subspace of $W^{1,p}(\Omega)$ formed by the functions with null mean in Ω . In this way, and motivated by physical motivations related with the wave equation we consider the conditions (F₁) – (F₂).

1 Introduction

In this paper we deal with the nonlocal elliptic system of the p-Kirchhoff type given by

$$(P) \begin{cases} - \left[M_1 \left(\int_{\Omega} |\nabla u|^p \right) \right]^{p-1} \Delta_p u = f(u, v) + \rho_1(x), & \Omega \\ - \left[M_2 \left(\int_{\Omega} |\nabla v|^p \right) \right]^{p-1} \Delta_p v = g(u, v) + \rho_2(x), & \Omega \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded smooth domain, $1 < p < N$, η is the unit exterior

vector on $\partial\Omega$, Δ_p is the p-Laplacian operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

and the involved functions in the problem satisfy:

$$(M) \begin{cases} M_1, M_2 : \mathbb{R}^+ \rightarrow \mathbb{R} \text{ are continuous functions} \\ \text{and there is a positive constant } m_0 > 0 \\ \text{such that } M_1(t), M_2(t) \geq m_0 > 0, \forall t \geq 0; \end{cases}$$

$$(F_1) \begin{cases} \text{There is a } C^1 \text{ function } F : \mathbb{R}^2 \rightarrow \mathbb{R} \text{ such that} \\ F_u(u, v) = f(u, v) \text{ and } F_v(u, v) = g(u, v) \\ \forall (u, v) \in \mathbb{R}^2; \end{cases}$$

There is $k > 0$ such that

$$(F_2) F(u + k, v + k) = F(u, v), \forall (u, v) \in \mathbb{R}^2;$$

$$(\rho) \begin{cases} \rho_1, \rho_2 \in L^q(\Omega), \frac{1}{p} + \frac{1}{q} = 1, 1 < q < p^* \text{ and} \\ \int_{\Omega} \rho_1 = \int_{\Omega} \rho_2 = 0. \end{cases}$$

Where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } 1 < p < N, \\ \infty & \text{if } p \geq N \end{cases}$$

is the critical Sobolev exponent.

Here $\int_{\Omega} h$ means $\int_{\Omega} h(x) dx$.

Using a variational method we will establish the following result:

Theorem 1.1. *Under assumptions (M), (F₁), (F₂) and (ρ), problem (P) possesses a weak solution $(u, v) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega)$.*

By a weak solution of (P) we mean a function $(u, v) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ such that

$$\left[M_1 \left(\int_{\Omega} |\nabla u|^p \right) \right]^{p-1} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi = \int_{\Omega} [f(u, v) + \rho_1] \varphi$$

and

$$\left[M_2 \left(\int_{\Omega} |\nabla v|^p \right) \right]^{p-1} \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \psi = \int_{\Omega} [g(u, v) + \rho_2] \psi$$

for all $\varphi, \psi \in W^{1,p}(\Omega)$.

System (P) is a generalization of a model introduced by Kirchhoff [5]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (1)$$

where ρ, P_0, h, E, L are constants, which extends the classical D'Alembert wave equation, by considering the effects of the changes in the length of the strings during the vibrations.

The equation

$$\frac{\partial^2 u}{\partial t^2} - M(\|u\|^2) \Delta u = f(u) \text{ in } \Omega \quad (2)$$

generalizes equation (1), where $M : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a given function and $\|u\|^2 = \int_{\Omega} |\nabla u|^2$ is the usual norm of $H_0^1(\Omega)$.

The stationary counterpart of (2) is

$$-M(\|u\|^2) \Delta u = f(x, u) \text{ in } \Omega \quad (3)$$

which, in the most of the papers, has been studied under Dirichlet boundary condition.

In the present work, besides the Neumann boundary condition, we consider the case of a system and, instead of the Laplacian, we work with the p -Laplacian.

This work is organized as follows: in section 2 we present some preliminary results and in the section 3 we prove the main result.

2 Preliminaries

We work essentially in the space $W^{1,p}(\Omega)$ endowed with the norm

$$\|u\|^p = \int_{\Omega} (|\nabla u|^p + |u|^p).$$

The norm in $L^p(\Omega)$ will be denoted by

$$|u|_p^p = \int_{\Omega} |u|^p.$$

We now set $X = W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ and $Y = L^p(\Omega) \times L^p(\Omega)$ which are Banach spaces equipped, respectively, by the norms

$$\begin{aligned} \|(u, v)\|_X &= \left(\int_{\Omega} (|\nabla u|^p + |\nabla v|^p) \right. \\ &\quad \left. + \int_{\Omega} (|u|^p + |v|^p) \right)^{1/p} \\ &= \left(\|u\|^p + \|v\|^p \right)^{1/p} \end{aligned}$$

and

$$|(u, v)|_Y = \left(\int_{\Omega} (|u|^p + |v|^p) \right)^{1/p} = (|u|_p^p + |v|_p^p)^{1/p}.$$

We remark that $W^{1,p}(\Omega)$ may be split in the following way. Let $W_c = \langle 1 \rangle$, that is, the subspace of $W^{1,p}(\Omega)$ spanned by the constant function 1 and $W_0 = \left\{ z \in W^{1,p}(\Omega); \int_{\Omega} z = 0 \right\}$ which is called the space of functions of $W^{1,p}(\Omega)$ with null mean in Ω . Thus

$$W^{1,p}(\Omega) = W_0 \oplus W_c,$$

i.e., every function $u \in W^{1,p}(\Omega)$ is of the form

$$u = u_0 + \alpha,$$

where $\int_{\Omega} u_0 = 0$ and α is a constant. Consequently, if $(u, v) \in X$, then

$$(u, v) = (u_0 + \alpha, v_0 + \beta) = (u_0, v_0) + (\alpha, \beta),$$

where $\int_{\Omega} u_0 = \int_{\Omega} v_0 = 0$ and α, β are constants.

As it is well known the Poincaré's inequality does not hold in the space $W^{1,p}(\Omega)$. However, it is true in W_0 as shows the next lemma for which we will give an alternative proof by using minimization on a certain manifold.

Lemma 2.1. (Poincaré-Wirtinger's Inequality) *There is a positive constant C such that*

$$\int_{\Omega} |z|^p \leq C \int_{\Omega} |\nabla z|^p, \text{ for all } z \in W_0.$$

Proof. Let $\psi : W_0 \rightarrow \mathbb{R}$ be the functional given by

$$\psi(z) = \int_{\Omega} |\nabla z|^p, \text{ for all } z \in W_0,$$

and M be the manifold

$$M = \left\{ z \in W_0; \int_{\Omega} |z|^p = 1 \right\}.$$

Since ψ is bounded from below on M , it follows that there is a minimizing sequence $(z_n) \subset M$, that is,

$$\psi(z_n) \rightarrow \inf_M \psi = \psi_0 \geq 0.$$

Consequently, $\int_{\Omega} |z_n|^p = 1$ and there is a positive constant C_1 such that $\int_{\Omega} |\nabla z_n|^p \leq C_1$, for all $n \in \mathbb{N}$. From these facts we infer that the sequence (z_n) is bounded in $W^{1,p}(\Omega)$ and in view of this, the real sequence $(\|z_n\|_p^p)$ possesses a convergent subsequence. So, $z_n \rightarrow z$ in $W^{1,p}(\Omega)$ and by virtue of compactness of the Sobolev embedding we have, perhaps for a subsequence, that $z_n \rightarrow z$ in $L^r(\Omega)$, $1 \leq r < p^*$, with p^* the critical Sobolev exponent. In particular, $0 = \int_{\Omega} z_n \rightarrow \int_{\Omega} z = 0$ and $1 = \int_{\Omega} |z_n|^p \rightarrow \int_{\Omega} |z|^p$ and thus $z \in M$.

Let us show that $\psi_0 > 0$. Suppose, on the contrary, that $\psi_0 = 0$. In this case, up to subsequences, we have

$$\begin{aligned} 0 &= \lim \int_{\Omega} |\nabla z_n|^p \\ &= \lim \left(\int_{\Omega} |\nabla z_n|^p + \int_{\Omega} |z_n|^p - \int_{\Omega} |z_n|^p \right) \\ &= \lim (\|z_n\|_p^p - |z_n|_p^p) \\ &= \lim \|z_n\|_p^p - \lim |z_n|_p^p \\ &\geq \|z\|_p^p - |z|_p^p = \int_{\Omega} |\nabla z|^p \end{aligned}$$

which yields

$$\int_{\Omega} |\nabla z|^p \leq 0.$$

Therefore, $z(x) = C$ a.e. in Ω , with C a real constant. Because $z \in M \subset W_0$, one has

$$\int_{\Omega} z = \int_{\Omega} C = 0$$

and we conclude that $C = 0$, which is impossible because $\int_{\Omega} |z|^p = 1$. Consequently, $\psi_0 > 0$. Thus,

$$\begin{aligned} \psi_0 &= \inf_M \psi \\ &= \lim \int_{\Omega} |\nabla z_n|^p = \lim (\|z_n\|_p^p - |z_n|_p^p) \\ &= \lim \|z_n\|_p^p - \lim |z_n|_p^p \geq \|z\|_p^p - |z|_p^p \\ &= \int_{\Omega} |\nabla z|^p \geq \psi_0. \end{aligned}$$

Hence,

$$\psi_0 = \int_{\Omega} |\nabla z|^p = \psi(z)$$

which shows that the infimum of ψ is attained on M . Consequently,

$$\psi_0 \leq \int_{\Omega} |\nabla z|^p, \text{ for all } z \in W_0 \text{ with } |z|_p = 1.$$

If $0 \neq z \in W_0$,

$$\psi_0 \leq \int_{\Omega} \left| \nabla \left(\frac{z}{|z|_p} \right) \right|^p = \frac{1}{|z|_p^p} \int_{\Omega} |\nabla z|^p$$

from which

$$|z|_p^p \leq \frac{1}{\psi_0} \int_{\Omega} |\nabla z|^p, \text{ for all } z \in W_0$$

which shows the Poincaré-Wirtinger's inequality in W_0 . \square

We will use the following version of the Ekeland Variational Principle [4] that will play a key role in the proof of our main result.

Proposition 2.2. *Let E be a Banach space and let $\psi : E \rightarrow \mathbb{R}$ be a C^1 function which is bounded from below. Then, for any $\epsilon > 0$ there exists $z \in E$ such that*

$$\psi(z) \leq \inf_E \psi + \epsilon \text{ and } \|\psi'(z)\|_{E^*} \leq \epsilon.$$

3 Proofs

Let us start by considering the functional $I : X \rightarrow \mathbb{R}$ given by

$$\begin{aligned} I(u, v) &= \frac{1}{p} \widehat{M}_1 \left(\int_{\Omega} |\nabla u|^p \right) + \frac{1}{p} \widehat{M}_2 \left(\int_{\Omega} |\nabla v|^p \right) \\ &\quad - \int_{\Omega} F(u, v) - \int_{\Omega} \rho_1 u - \int_{\Omega} \rho_2 v, \end{aligned}$$

for all $(u, v) \in X$, where

$$\widehat{M}_i(t) = \int_0^t [M_i(s)]^{p-1} ds, \quad i = 1, 2.$$

Proposition 3.1. *The functional I is bounded from below.*

Proof. *Firstly, we note that I is well defined. For, it is enough to show that*

$$J: X \rightarrow \mathbb{R}$$

$$(u, v) \mapsto J(u, v) = \int_{\Omega} F(u, v)$$

is well defined. Since F is continuous on $[0, k] \times [0, k]$ and $F(u+k, v+k) = F(u, v)$ for all $(u, v) \in \mathbb{R}^2$, it follows that $|F(u, v)| \leq C$, for all $(u, v) \in \mathbb{R}^2$, and so

$$|J(u, v)| \leq \int_{\Omega} |F(u, v)| \leq C|\Omega|, \text{ for all } (u, v) \in X,$$

where $|\Omega|$ is the Lebesgue measure of Ω .

Let us show that I is bounded from below. If $(u, v) \in X$, u and v may be written as

$$u = u_0 + \alpha \text{ and } v = v_0 + \beta,$$

where $\alpha, \beta \in \mathbb{R}$ and $\int_{\Omega} u_0 = \int_{\Omega} v_0 = 0$. Thus

$$I(u, v) = \frac{1}{p} \widehat{M}_1 \left(\int_{\Omega} |\nabla u|^p \right) + \frac{1}{p} \widehat{M}_2 \left(\int_{\Omega} |\nabla v|^p \right)$$

$$- \int_{\Omega} F(u_0 + \alpha, v_0 + \beta)$$

$$- \int_{\Omega} \rho_1(u_0 + \alpha) - \int_{\Omega} \rho_2(v_0 + \beta)$$

$$\geq \frac{1}{p} \widehat{M}_1 \left(\int_{\Omega} |\nabla u|^p \right) + \frac{1}{p} \widehat{M}_2 \left(\int_{\Omega} |\nabla v|^p \right)$$

$$- C|\Omega| - \int_{\Omega} \rho_1(u_0 + \alpha) - \int_{\Omega} \rho_2(v_0 + \beta)$$

$$\geq \frac{1}{p} \widehat{M}_1 \left(\int_{\Omega} |\nabla u|^p \right) + \frac{1}{p} \widehat{M}_2 \left(\int_{\Omega} |\nabla v|^p \right)$$

$$- C|\Omega| - \int_{\Omega} \rho_1 u_0 - \alpha \int_{\Omega} \rho_1$$

$$- \int_{\Omega} \rho_2 v_0 - \beta \int_{\Omega} \rho_2.$$

Since $\rho_1, \rho_2 \in L^q(\Omega)$, $u_0, v_0 \in L^p(\Omega)$ and using the Hölder inequality

$$\int_{\Omega} \rho_1 u_0 \leq |\rho_1|_q |u_0|_p \text{ and } \int_{\Omega} \rho_2 v_0 \leq |\rho_2|_q |v_0|_p$$

we obtain

$$I(u, v) \geq \frac{1}{p} \widehat{M}_1 \left(\int_{\Omega} |\nabla u|^p \right) + \frac{1}{p} \widehat{M}_2 \left(\int_{\Omega} |\nabla v|^p \right)$$

$$- |\rho_1|_q |u_0|_p - |\rho_2|_q |v_0|_p - C|\Omega|$$

and using assumption (M)

$$I(u, v) \geq \frac{1}{p} m_0^{p-1} \left(\int_{\Omega} |\nabla u|^p \right) + \frac{1}{p} m_0^{p-1} \left(\int_{\Omega} |\nabla v|^p \right)$$

$$- |\rho_1|_q |u_0|_p - |\rho_2|_q |v_0|_p - C|\Omega|.$$

By means of Poincaré-Wirtinger's inequality

$$I(u, v) \geq \frac{1}{p} m_0^{p-1} \left(\int_{\Omega} |\nabla u|^p + \int_{\Omega} |\nabla v|^p \right)$$

$$- \frac{1}{\psi_0^{1/p} |\rho_1|_q} \left(\int_{\Omega} |\nabla u_0|^p \right)^{1/p}$$

$$- \frac{1}{\psi_0^{1/p} |\rho_2|_q} \left(\int_{\Omega} |\nabla v_0|^p \right)^{1/p} - C|\Omega|$$

Because the function

$$(s, t) \mapsto \frac{1}{p} m_0^{p-1} (s + t) - \frac{1}{\psi_0^{1/p} |\rho_1|_q} s^{1/p}$$

$$- \frac{1}{\psi_0^{1/p} |\rho_2|_q} t^{1/p}, \quad s, t \geq 0,$$

is bounded from below, we conclude that I is also bounded from below. \square

Remark 3.2. By this last proposition, the main difference between the Dirichlet and Neumann problem is related with the noncoerciveness of the energy functional associated to Neumann problem. At this point the Poincaré-Wirtinger's inequality plays a fundamental role. The reader may consult [1], [6], [7] where some Kirchhoff equations are considered, from the variational point of view, under Dirichlet boundary conditions in the scalar case. Some classes of M and nonlinearities f are considered in problems of the type

$$\begin{cases} - \left[M(\|u\|_{1,p}^p) \right]^{p-2} \Delta_p u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\|u\|_{1,p}^p = \int_{\Omega} |\nabla u|^p$ is the usual norm in $W_0^{1,p}(\Omega)$.

Proof of Theorem 1.1 *We will find a critical point of the functional I . As I is C^1 functional bounded from below, it follows from the Ekeland Variational Principle that there exists $((u_n, v_n)) \subset X$ such that*

$$I(u_n, v_n) \rightarrow \inf_X I \text{ and } I'(u_n, v_n) \rightarrow 0. \quad (4)$$

For each $n \in \mathbb{N}$,

$$u_n = u_n^0 + \alpha_n \quad \text{and} \quad v_n = v_n^0 + \beta_n,$$

where α_n and β_n are real constants and $\int_{\Omega} u_n^0 = \int_{\Omega} v_n^0 = 0$. From (4) we have $|I(u_n, v_n)| \leq C_1$, for some positive constant C_1 and for all $n \in \mathbb{N}$. We now use the preceding proposition to obtain

$$\begin{aligned} C_2 &\leq \frac{1}{p} m_0^{p-1} \left(\int_{\Omega} |\nabla u_n^0|^p + \int_{\Omega} |\nabla v_n^0|^p \right) \\ &\quad - \frac{1}{\psi_0^{1/p} |\rho_1|_q} \left(\int_{\Omega} |\nabla u_n^0|^p \right)^{1/p} \\ &\quad - \frac{1}{\psi_0^{1/p} |\rho_2|_q} \left(\int_{\Omega} |\nabla v_n^0|^p \right)^{1/p} \\ &\leq I(u_n, v_n) \leq C_1 \end{aligned}$$

which implies that the sequences $\left(\int_{\Omega} |\nabla u_n^0|^p \right)$ and $\left(\int_{\Omega} |\nabla v_n^0|^p \right)$ are bounded. By virtue of the Poincaré inequality $\left(\int_{\Omega} |u_n^0|^p \right)$ and $\left(\int_{\Omega} |v_n^0|^p \right)$ are bounded too. Consequently, (u_n^0) and (v_n^0) are bounded sequences in $W^{1,p}(\Omega)$. Remarking that α_n, β_n may be taken in the interval $[0, k]$, $u_n = u_n^0 + \alpha_n$ and $v_n = v_n^0 + \beta_n$, for all $n \in \mathbb{N}$, the sequences (u_n) and (v_n) are bounded in $W^{1,p}(\Omega)$. Hence,

$$u_n \rightharpoonup \bar{u} \quad \text{and} \quad v_n \rightharpoonup \bar{v} \quad \text{in} \quad W^{1,p}(\Omega), \quad (5)$$

perhaps for subsequences, and so

$$\int_{\Omega} \rho_1 u_n \rightarrow \int_{\Omega} \rho_1 \bar{u}, \quad \int_{\Omega} \rho_2 v_n \rightarrow \int_{\Omega} \rho_2 \bar{v}$$

and thanks to Sobolev embeddings

$$u_n \rightarrow \bar{u} \quad \text{and} \quad v_n \rightarrow \bar{v} \quad \text{in} \quad L^s(\Omega), \quad 1 \leq s < p^*,$$

where p^* is the critical Sobolev exponent, and up to subsequences,

$$u_n(x) \rightarrow \bar{u}(x) \quad \text{and} \quad v_n(x) \rightarrow \bar{v}(x) \quad \text{a.e. in } \Omega.$$

Due to the continuity of F

$$F(u_n(x), v_n(x)) \rightarrow F(\bar{u}(x), \bar{v}(x)) \quad \text{a.e. in } \Omega$$

and because $|F(u_n(x), v_n(x))| \leq C$ for all $n \in \mathbb{N}$ a.e. in Ω , we may use the Lebesgue dominated convergence theorem to conclude that

$$\int_{\Omega} F(u_n, v_n) \rightarrow \int_{\Omega} F(\bar{u}, \bar{v}). \quad (6)$$

Recalling that

$$\begin{aligned} \inf_X I &= \lim I(u_n, v_n) \\ &= \lim \left[\frac{1}{p} \widehat{M}_1 \left(\int_{\Omega} |\nabla u_n|^p \right) \right. \\ &\quad \left. + \frac{1}{p} \widehat{M}_2 \left(\int_{\Omega} |\nabla v_n|^p \right) \right] \\ &\quad - \int_{\Omega} F(u_n, v_n) - \int_{\Omega} \rho_1 u_n - \int_{\Omega} \rho_2 v_n \end{aligned}$$

it rests to analyze the convergence of $\left(\widehat{M}_1 \left(\int_{\Omega} |\nabla u_n|^p \right) \right)$ and $\left(\widehat{M}_2 \left(\int_{\Omega} |\nabla v_n|^p \right) \right)$.

As $u_n \rightharpoonup \bar{u}$ in $W^{1,p}(\Omega)$ one has that

$$\int_{\Omega} |u_n|^p \rightarrow \int_{\Omega} |\bar{u}|^p$$

and

$$\|\bar{u}\|^p \leq \liminf \left(\int_{\Omega} |\nabla u_n|^p + \int_{\Omega} |u_n|^p \right). \quad (7)$$

Since $\left(\int_{\Omega} |\nabla u_n|^p \right)$ is bounded, it is convergent, at least for a subsequence. Hence, from (7),

$$\|\bar{u}\|^p \leq \lim \int_{\Omega} |\nabla u_n|^p + \lim \int_{\Omega} |u_n|^p$$

which implies

$$\int_{\Omega} |\nabla \bar{u}|^p + \int_{\Omega} |\bar{u}|^p \leq \lim \int_{\Omega} |\nabla u_n|^p + \int_{\Omega} |\bar{u}|^p.$$

Thus,

$$\int_{\Omega} |\nabla \bar{u}|^p \leq \lim \int_{\Omega} |\nabla u_n|^p.$$

Noticing that \widehat{M}_1 and \widehat{M}_2 are continuous and increasing, we get

$$\widehat{M}_1 \left(\int_{\Omega} |\nabla \bar{u}|^p \right) \leq \lim \widehat{M}_1 \left(\int_{\Omega} |\nabla u_n|^p \right).$$

Analogously,

$$\widehat{M}_2 \left(\int_{\Omega} |\nabla \bar{v}|^p \right) \leq \lim \widehat{M}_2 \left(\int_{\Omega} |\nabla v_n|^p \right).$$

Consequently,

$$\begin{aligned} \inf_X I &\geq \frac{1}{p} \widehat{M}_1 \left(\int_{\Omega} |\nabla \bar{u}|^p \right) + \frac{1}{p} \widehat{M}_2 \left(\int_{\Omega} |\nabla \bar{v}|^p \right) \\ &\quad - \int_{\Omega} F(\bar{u}, \bar{v}) - \int_{\Omega} \rho_1 \bar{u} - \int_{\Omega} \rho_2 \bar{v} = I(\bar{u}, \bar{v}) \end{aligned}$$

that implies that $I(\bar{u}, \bar{v}) = \inf_X I$. Since $(\bar{u}, \bar{v}) \in W^{1,p}(\Omega)$ and is a weak solution of problem (P) we conclude that such a function satisfies the Neumann boundary condition in the trace sense. This finishes the proof of the theorem. \square

Remark 3.3. If one considers the Hamiltonian system

$$\begin{cases} - \left[M_1 \left(\int_{\Omega} |\nabla u|^p \right) \right]^{p-1} \Delta_p u = f(v) + \rho_1(x), \Omega \\ - \left[M_2 \left(\int_{\Omega} |\nabla v|^p \right) \right]^{p-1} \Delta_p v = g(u) + \rho_2(x), \Omega \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \partial\Omega. \end{cases} \quad (8)$$

where $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are P-periodic and continuous functions satisfying

$$\int_0^P f(s) ds = \int_0^P g(s) ds = 0$$

we can prove, reasoning as in the theorem 1.1, the existence of a weak solution of the problem (P), since M, ρ_1, ρ_2 satisfy assumptions (M) and (ρ).

The situation described in the problem (8) is the counterpart, in nonlocal partial differential equations setting, of that considered in Willem [8] for the forced pendulum equation. Also related with this kind of problem the reader may consult Willem [9] and Brézis [2].

Note that these assumptions imply that the involved functions f, g, ρ_1, ρ_2 change sign. Thus we can not expect to obtain positive solutions.

Remark 3.4. As it is easy to see the condition (ρ) is necessary in the (M_1, M_2) – linear case, that is,

$$\begin{cases} - \left[M_1 \left(\int_{\Omega} |\nabla u|^p \right) \right]^{p-1} \Delta_p u = \rho_1(x), \Omega \\ - \left[M_2 \left(\int_{\Omega} |\nabla v|^p \right) \right]^{p-1} \Delta_p v = \rho_2(x), \Omega \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \partial\Omega. \end{cases} \quad (9)$$

Thus, it is natural to consider this assumption in the semilinear case.

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