Some notions in non associatives genetics algebras

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April 14 2008

Resumo: *Non-associative algebras is a branch of Algebra that has many applications. In particular, t-algebras have applications in Genetics. The main purpose of this work is to introduce, for t-algebras of rank 3, the notions of exceptional, normal and cancelative t-algebras of rank 3 (by analogy with Bernstein algebras). In sequence, some results are proved.*

0 Introduction

Addiction of vectors and multiplication of vectors by scalars, or convex combinations, describes mixtures of populations. Furthermore, a multiplicative structure correspond to a sexual reproduction, and so, algebras occur naturally in population genetic models. The basic law

$$xy \times xy = \frac{1}{4}xx + \frac{1}{2}xy + \frac{1}{4}yy$$

sugests a formal algebric representation of genetics roles. In addiction to the algebraic structure, some models have others caractheristic properties.

In the period from 1939 to 1951, I.M.H. Etherington writed some articles in which he introduced many new concepts in non-associative algebras, in order to construct models for the studying of Genetic of Populations (see [4], [5] and [6]). He started the branch of genetic algebras giving a precise mathematical formulation of Mendel´s law, introducing the notions of baric algebras and train algebras and showing that several concrete biological situations are instances of their. In particular, he considered the t-algebras, that are non-associatives algebras, with a nonzero homomorphism ω in the scalar field and satisfying a polinomial equation which involves the homomorphism ω. Now the fundamental results are known and it is possible to evaluate the difficulty to classify these algebras. The article [9] contains a critical analysis of the basic concepts of the algebras theory that are important in Genetic of Populations. Specially, it contains the classical material about the t-algebras, as well as the motivation for its introduction by Etherington.

The great interest of the science in genetic questions in the last decades, and the response of Algebra with the works of Etherington ([4], [5], [6]), Holgate ([8]), Schafer ([10]) and others opened a new way of investigation about this algebraic structures, whose properties are being discovered and applied in a increasing numbers of situations. In this paper, we introduce some notions with the aim of adding models which must be applied in new situations.

1 Preliminaries

Let F be a field with $\mathrm{char}(F) \neq 2$ and A be an F-algebra, not necessarily associative. If $\omega : A \to F$ is a nonzero homomorphism, then the ordered pair (A, ω) is called a *baric algebra* over F and ω its *weight function.* For each $x \in A$, $\omega(x)$ is called the *weight of x.* Let (A, ω) be a commutative baric algebra. If there exist γ_1, $\gamma_2, \ldots, \gamma_{n-1} \in F$ such that, for all x in A,

$$x^n + \gamma_1\omega(x)x^{n-1} + \ldots + \gamma_{n-1}\omega(x)^{n-1}x = 0 \quad (1)$$

we say that (A, ω) is a *(commutative) train algebra* (in short, *t-algebra*). Moreover, if there is no similar relation involving $x^{n-1}, \ldots, x,$

we say that n is the rank of (A, ω) and (1) is its *train equation* (in short, *t-equation*). In this case, it is easy to see that $1 + \gamma_1 + \gamma_2 + \ldots + \gamma_{n-1} = 0$. Moreover, $x^n = 0$ for all $x \in Ker(\omega)$. If (A, ω) is a commutative t-algebra of rank n, then its t-equation is unique. In this paper, we will consider only t-algebras of rank 3, that is, those satisfying an equation

$$x^3 - (1 + \gamma)\omega(x)x^2 + \gamma\omega(x)^2 x = 0 \quad (2)$$

where $\gamma \in F$ is a fixed element. For details about t-algebras of rank 3, the reader is referred to [2], [3] and [11]. In this paper we assume that $2\gamma \neq 1$ and, as a consequence, there is (at least) an idempotent $e \in A$ and relative to this element, A has a Peirce decomposition $A = Fe \oplus U_e \oplus V_e$ in which, denoting $Ker(\omega)$ by N,

$$U_e = \{u \in N : 2eu = u\} \quad (3)$$

$$V_e = \{v \in N : ev = \gamma v\} \quad (4)$$

$$N = U_e \oplus V_e \quad (5)$$

The decomposition $A = Fe \oplus U_e \oplus V_e$ depends on the choice of the idempotent in A, but it can be proved that the dimensions of U_e and V_e are invariants, see [1]. Then we can define the invariant *type of* A as the ordered pair of integers $(1 + r, s)$, where $r = dim(U_e)$ and $s = dim(V_e)$. The subspaces U_e and V_e satisfy the following relations: $U_e^2 \subseteq V_e$; $U_e V_e \subseteq U_e$; $V_e^2 = 0$; $U_e^{2n+1} \subseteq U_e$ $(n \geq 0)$ and $U_e^{2n} \subseteq V_e$ $(n \geq 1)$. Moreover, by the second linearization of (2), for all elements $x, y, z \in N$, we obtain the Jacobi's identity

$$J(x, y, z) = x(yz) + y(xz) + z(xy) = 0 \quad (6)$$

Let $I(A)$ be the set of the idempotents elements of A. It follows of the above relations that $I(A) = \{e_0 = e + u_0 + \lambda u_0^2 : u_0 \in U_e\}$, where $\lambda = (1 + - 2\gamma)^{-1}$. We consider the Peirce decomposition of A in relation to the idempotent e_0, that is, $A = Fe_0 \oplus U_0 \oplus V_0$, where $U_0 = \{u \in N : 2e_0 u = u\}$ and $V_0 = \{v \in N : e_0 v = \gamma v\}$. The subspaces U_e an U_0, V_e and V_0 are related by $U_0 = \{u + 2\lambda\ u_0 u : u \in U_e\}$ and $V_0 = \{v - 2\lambda\, u_0 v : v \in V_e\}$.

Now we will explore a result that appears shortly mentioned in [7]. For that, consider $A = Fe \oplus U_e \oplus V_e$ a t-algebra of rank 3. In order to simplify notation, we use U and V in place of U_e and V_e, respectively. To each fixed $\gamma \in F$, $2\gamma \neq 1$, there is a class of t-algebras of rank 3 that satisfy the equation $x^3 - (1 + \gamma)\omega(x)x^2 + \gamma\omega(x)x = 0$. If A satisfy the equation $x^3 - \omega(x)x^2 = 0$ (that is, if $\gamma = 0$), it is possible to pass to other class in which $\gamma \neq 0$ if we define over the F-vectorial space A a new multiplication as follows:

$$x \circ y = (1 - 2\gamma)xy + \gamma[x\omega(y) + y\omega(x)] \quad (7)$$

with $\gamma \in F$ different from $\frac{1}{2}$. We have the following results about this new algebra (which will be denoted by A_1):

Lemma 1.1 Let $A = Fe \oplus U \oplus V$ be a t-algebra of rank 3 that satisfies the t-equation $x^3 - \omega(x)x^2 = 0$. Consider $\gamma \in F$, with $2\gamma \neq 1$ and A_1 the algebra defined as above. Then

(a) A_1 is baric, commutative, with weight function ω;

(b) A and A_1 have the same idempotents;

(c) A_1 satisfies the t-equation $x^3 - (1 + \gamma)\omega(x)x^2 + \gamma\omega(x)^2 x = 0$;

(d) The Peirce decomposition of A_1, relative to the idempotent e, is $A_1 = = Fe \oplus U \oplus V$.

Proof: To prove (a) and (c), see [7].

(b) If $e \in A$ is an idempotent element, then $e \circ e = (1 - 2\gamma)e^2 + 2\gamma\omega(e)e = = (1 - 2\gamma)e + 2\gamma e = e$. Conversely, $e \circ e = e$ implies $(1 - 2\gamma)e^2 = e \circ e - 2\gamma e = = (1 - 2\gamma)e$, that is, $e^2 = e$.

(d) Consider $U^* = \{u \in N : 2\ e \circ u = u\}$ and $V^* = \{v \in N : e \circ v = \gamma v\}$. If $u \in \in U^*$, then $u = u_1 + v_1$, with $u_1 \in U$, $v_1 \in V$ and so $u = 2\ e \circ u = 2\ e \circ (u_1 + v_1) = 2\ e \circ u_1 + 2\ e \circ v_1 = 2[(1 - 2\gamma)eu_1 + \gamma u_1] + 2[(1 - 2\gamma)ev_1 + \gamma v_1] = (1 - 2\gamma)u_1 + + 2\gamma u_1 + 2\gamma v_1 = (1 - 2\gamma)u_1 + 2\gamma u$. Therefore, $(1 - 2\gamma)u = (1 - 2\gamma)u_1$ and this means that $u = u_1$ and so $U^* \subseteq U$.

Conversely, if $u \in U$, then $2e \circ u = 2[(1-2\gamma)eu + \gamma u] = 2eu = u$ so $u \in U^*$ and this implies $U^* = U$. In the same way it can be proved that $V^* = V$. ■

2 Definitions and Examples

In the Bernstein algebras theory, there are known the subclasses of the *nuclear*, *exceptional*, *normal* and *cancelative algebras*. By analogy, we extend those ideas to t-algebras of rank 3. For that, consider $A = Fe \oplus U \oplus V$ a t-algebra of rank 3, with $\gamma \neq 0$. Observe that $A^2 = A$, since if $x = ke + u + v \in A$, and so $x = ke^2 + 2eu + \gamma^{-1}ev \in A^2$ and it is not necessary to define nuclear t-algebras of rank 3. In the following, we use $< w_1, w_2, \ldots, w_n >$ to denote the vectorial subspace generated by the vectors $w_1, w_2, \ldots, w_n$ in some vectorial space.

Definition 2.1 A t-algebra of rank 3 (A, ω) satisfying (2) is called a
(i) *exceptional t-algebra of rank 3* if $U^2 = 0$, for some Peirce decomposition of A;
(ii) *normal t-algebra of rank 3* if $UV = 0$, for some Peirce decomposition of A.

Example A simple computation shows that the algebra $A_1 = < e, u_1, u_2, v_1, v_2 >$, which multiplication table is $e^2 = e$, $2eu_i = u_i$ $(i = 1,2)$, $ev_i = v_i$ $(i = 1,2)$, $u_1v_2 = u_2$ and the other products are zero is a exceptional t-algebra of rank 3. Moreover, the algebra $A_2 = < e, u, u^2, v >$, which multiplication table is $e^2 = e, 2eu = u, u^2 \neq 0$ and the other products are zero is a t-algebra of rank 3, but A_2 is not a exceptional one. In the same way, A_2 is a normal t-algebra of rank 3 and A_1 is a t-algebra of rank 3, but is not a normal one.

Lemma 2.2 Let $A = Fe \oplus U \oplus V$ be a t-algebra of rank 3 satisfying (2). Then

(i) If A is a exceptional t-algebra, then A is a exceptional t-algebra, relate to any idempotent element $e_0 \in I(A)$;

(ii) If A is a normal t-algebra, then A is a normal t-algebra, relate to any idempotent element $e_0 \in I(A)$.

Proof: (i) Since $e_0 \in I(A)$, there is $u_0 \in U$ such that $e_0 = e + u_0 + \lambda u_0^2$ and so $U_0 = \{u + 2\lambda u_0 u : u \in U\} = U$, as $u_0 u \in U^2 = 0$.

(ii) It is enough to see that, if $e_0 = e + u_0 + \lambda u_0^2 \in I(A)$, then a generical generator of U_0V_0 is $(u + 2\lambda u_0 u)v = uv + 2\lambda(u_0u)v = uv = 0$, since $(u_0u)v \in V^2 = 0$. ■

The next Proposition gives a characterization for normal t-algebras of rank 3.

Proposition 2.3 $A = Fe \oplus U \oplus V$ is a normal t-algebra of rank 3 satisfying (2), then for all $x, y \in A$

$$x^2y - \omega(x)xy - \gamma\,\omega(y)[x^2 - \omega(x)x] = 0 \quad (8)$$

Proof: Let $x = ke + u + v$ and $y = k'e + u' + v'$ be two arbitrary elements in a normal t-algebra of rank 3 A. It is easy to see that

$$2x^2 = 2k^2e + 2ku + 2u^2 + 4\gamma kv$$

$$2xy = 2kk'e + k'u + ku' + 2\gamma k'v + 2\gamma kv' + 2uu'$$

$$2x^2y = 2k^2k'e + kk'u + k^2u' + 2\gamma k'u^2 + 4\gamma^2 kk'v +$$

$$+2kuu' + 2\gamma k^2 v'$$

and so, by a simple computation, we prove (8). Conversely, consider $uv \in UV$ and let, in (8), $x = u + v$ and $y = e$. Then

$$(u+v)^2e - \omega(u+v)[(u+v)e] - \gamma\omega(e)[(u+v)^2$$

$$-w(u+v)(u+v)] = 0$$

By $\omega(u+v) = 0$, we have

$$(u^2 + 2uv + v^2)e - \gamma(u^2 + 2uv + v^2) = 0$$

and, by $eu^2 = \gamma u^2$, $2e(uv) = uv$ and $v^2 = 0$, it follows that

$$\gamma u^2 + uv - \gamma u^2 + 2\gamma uv = 0$$

That is, $(1-2\gamma)uv = 0$ and by $1 - 2\gamma \neq 0$, we have $uv = 0$. ■

Definition 2.4 A t-algebra of rank 3 $A = Fe \oplus U \oplus V$ is called a *cancelative* t-algebra if, for all $0 \neq u \in U$, the linear application $L_{\mathrm{u}} : U \longrightarrow V$ defined by $L_{\mathrm{u}}(u_1) = uu_1$, is a injective application.

The algebra of Example A_2 is a cancelative t-algebra of rank 3. Consider now the algebra $A_3 =< e, u_1, u_2, u_1u_2, v >$, which multiplication table is $e^2 = e$, $2eu_i = u_i$ $(i = 1, 2)$, $u_1u_2 \neq 0$ and the other products are zero. Then A_3 is a t-algebra of rank 3 (satisfying the t-equation $x^3 - \omega(x)^2x = 0$). Moreover, L_{u_1} is not injective, since $u_1 \neq 0$ and, for example, $L_{\mathrm{u}_1}(u_1) = 0$ and so A_3 is not a exceptional t-algebra of rank 3.

Proposition 2.5 If $A = Fe \oplus U \oplus V$ is a cancelative t-algebra of rank 3, then A is a normal t-algebra of rank 3.

Proof: Let $0 \neq u \in U$ and $v \in V$ be arbitrary elements. In the the Jacobi's identitie, by considering $x = y = u$ and $z = v$, we have $2u(uv) + u^2v = 0$ and since $u^2v \in V^2 = \{0\}$, it follows that $L_u(uv) = u(uv) = 0$. But A is cancelative t-algebra and $u \neq 0$, and so $uv = 0$. By linearity, we get $UV = 0$ and so A is normal. ■

Consider now the algebra $A_4 = < e, u_1, u_2, u_1^2, u_2^2 >$, which multiplication table is $e^2 = e, 2eu_i = u_i$ $(i = 1, 2), u_i^2 \neq 0$ $(i = 1, 2)$, and the other products are zero. It is easy to prove that A_4 is t-algebra of rank 3, satisfying the t-equation $x^3 - \omega(x)^2x = 0$. By the multiplication table, we see that $UV = 0$ and so A_4 is a normal t-algebra. Moreover, $u_i \neq 0$, for $i = 1, 2$, and $L_{u_1}(u_2) = u_1u_2 = 0$, that is, A_4 is not a cancelative t-algebra.

Lemma 2.6 If $A = Fe \oplus U \oplus V$ is a cancelative t-algebra of rank 3, then A is a cancelative t-algebra, relate to any idempotent element $e_0 \in I(A)$.

Proof: Let $e_0 = e + u_0 + \lambda u_0^2 \in I(A)$ be a idempotent element of A and U_0 the corresponding Peirce's subspace. Consider now $0 \neq u' = u + \lambda u_0 u$ and $u_1' = u_1 + \lambda u_0 u_1 \in U_0$ such that $L_{u'}(u_1') = u'u_1' = 0$. Since A is a cancelative algebra, by Proposition 2.5, A is a normal one and so $u_0(uu_1) \in UV = 0$. Then $0 = (u + \lambda u_0 u)(u_1 + \lambda u_0 u_1) = uu_1 = L_u(u_1)$ and so $u_1 = 0$, which implies $u_1' = 0$. ■

3 Some results about cancelatives t-algebras of rank 3

A Jordan algebra is a commutative nonassociative algebra A that satisfies the identity $(x^2y)x - x^2(yx) = 0$, for all $x, y \in A$.

Theorem 3.1 If A is a cancelative t-algebra of rank 3 satisfying the train equation (2), then the following conditions are equivalents:

(i) A is a Jordan algebra;

(ii) $\gamma = 0$ or $\gamma = 1$.

Proof: By Proposition 2.5, A is a normal algebra and, by Proposition 2.3, satisfies (8) and so, $(x^2y)x = \omega(x)x(xy) + \gamma\, \omega(y)[x^3 - \omega(x)x^2]$. By linearizing this last identity, we obtain

$$2(xz)y - \omega(z)xy - \omega(x)yz$$

$$-\gamma\omega(y)[2xz - \omega(z)x - \omega(x)z] = 0 \quad (9)$$

and for $x, z = x$ and $y = yx$, we have

$$x^2(yx) = \omega(x)x(xy) + \gamma\omega(xy)[x^2 - \omega(x)x]$$

Now, we can prove the equivalences. For that, consider $x, y \in A$. Then $(x^2y)x = x^2(yx)$ if, and only if,

$$\gamma\, \omega(y)[x^3 - \omega(x)x^2] = \gamma\omega(xy)[x^2 - \omega(x)x] \Leftrightarrow$$

$$\Leftrightarrow \gamma\, \omega(y)[x^3 - 2\omega(x)x^2 + \omega(x)^2x] = 0$$

Consider, now, $y \in A$ such that $\omega(y) = 1$. Then $(x^2y)x = x^2(yx) \iff \gamma[x^3 - 2\omega(x)x^2 + \omega(x)^2x] = 0 \iff \gamma = 0$ or $x^3 - 2\omega(x)x^2 + \omega(x)^2x = 0 \iff \gamma = 0$ or $\gamma = 1$. ■

Proposition 3.2 If A is a cancelative t-algebra of rank 3 of type $(1 + r, s)$, then $0 \leq r \leq s$.

Proof: If $r = 0$, then the result is valid. Suppose, now, $r > 0$. Then $U \neq 0$ and we can consider $0 \neq u \in U$. Since A is a cancelative algebra, the linear application $L_u : U \longrightarrow V$ is injective and so $r = dim(U) \leq dim(V) = s$. ■

Theorem 3.3 Let A be a t-algebra of rank 3 of type $(1 + r, s)$, $r \geq 1$, satisfying

$2dim(U^2) = r(r+1)$. Then A is a cancelative t-algebra.

Proof: Consider $0 \neq u \in U$ and $u_0 \in U$ such that $uu_0 = 0$. Let $B_U = \{u = = u_1, u_2, \ldots, u_r\}$ be a basis of U. Then $U^2 = [u_i u_j]$, for $1 \leq i, j \leq r$. Since A is a commutative algebra, there are $\frac{1}{2}r(r+1) = dim(U^2)$ distinct elements of type $u_i u_j$, for $1, \leq i, j \leq r$ and so these elements form a basis for U^2. Otherwise, $u_0 \in U$ and so there are $\alpha_1, \alpha_2, \ldots, \alpha_r$ such that

$$u_0 = \sum_{i=1}^{r} \alpha_i u_i$$

and so

$$0 = uu_0 = u_1 u_0 = \sum_{i=1}^{r} \alpha_i u_1 u_i$$

Since the elements $u_1 u_i (1 \leq i \leq r)$ are free, it follows that $\alpha_i = 0$ (for $i = 1, 2, \ldots, r$) and so $u_0 = 0$. Then L_u is a injective application, for all $0 \neq u \in U$; that is, A is a cancelative algebra. ■

4 Conclusion

We think Biomathematics as an application of mathematical methods and models to understand biological phenomenons. For Genetic of Populations, a subarea of Biomathematics, are interestings the situations in which perturbations of the Hardy-Weinberg Principle occur, that is, genetics frequences variations, and its consequent inheritance propagation, produced for example by mutations, as in the selective breeding with a non random crossing. The little contribuition of this work is the presentation of notions and initials results about a particular type of baric algebras, the train-algebras, producing models that must be used to describe, to understand and to control, these situations. For that, in this first work we introduced some classes of t-algebras of rank 3 (exceptional, normal and cancelative) and we proved some results for them. A central problem in Algebra is "classification". Now, we are able to use these results for trying to classify the t-algebras of rank 3 and dimension 6, of type (4,2), and this will complete the classsification of t-algebras of rank 5 and dimension 6.

Referências

[1] R. Arbach; L.A.O. Fernandes, Subclasses of a train-algebra of rank 3, *Revista Brasileira de Biometria*, 25, n. 1, (2007) 23-29.

[2] R. Arbach, Sobre os P-subespaços em uma train álgebra de posto 3, Thesis (Ph.D.) - Instituto de Matemática e Estatística da Universidade de São Paulo (1997) 136p.

[3] R. Costa, Principal t-algebras of rank 3 and dimension ≤ 5, *Proc. Edinb. Math. Soc.* 33, (1990)61-70.

[4] I.M.H. Etherington, On non-associative combinations, *Proceedings Royal Society Edinburgh.* 59 (1939) 153-162.

[5] I.M.H. Etherington, Genetic Algebras, *Proc. Roy. Soc. Edinburgh* 59 (1939) 242-258.

[6] I.M.H. Etherington, Non-commutative train algebras of ranks 2 and 3, *Proceedings of the London Mathematical Society (2)* 52 (1951) 241-252.

[7] H. Guzzo Jr.; P. Vicente, On Bernstein and train algebras of rank 3, *Comm. Algebra* 26, no. 7, (1998) 2021-2032.

[8] P. Holgate, Genetic Algebras associated with polyploidy *Proc. Edinburgh Math. Soc. (2)*, 15 (1966) 1-9, Corrigendum ibid. 17 (1970) 120.

[9] M.L. Reed, Algebraic Structure of Genetic Inheritance, *Bulletin (New Series) of the American Mathematical Society* 34, n. 2 (1997) 107-130.

[10] R.D. Schafer, Structure of Genetic Algebras, *Americam Journal of Math.*, 71 (1949) 121-135.

[11] A. Wőrz, Algebras in Genetics, *Lecture Notes in Biomathematics* 36, Berlin-Heidelberg-New York, (1980) 237p.