

Numerical solution of hyperbolic-elliptic systems of conservation laws by multiresolution schemes

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Abstract

In this work we examine initial value problems for first-order systems of conservation laws

$$\begin{aligned}\partial_t \phi_1 + \partial_x f_1(\phi_1, \phi_2) &= 0, \\ \partial_t \phi_2 + \partial_x f_2(\phi_1, \phi_2) &= 0,\end{aligned}\tag{1}$$

by applying WENO multiresolution schemes [8].

High resolution schemes that have been applied to multi-species kinematic models include WENO schemes [23] and central difference schemes, such as the Kurganov-Tadmor [20] and Nessyahu-Tadmor [21] schemes [4]. In [8] we demonstrated that the multiresolution technique applied to WENO schemes together with a sparse point representation (SPR) leads to an efficient and accurate scheme for multi-species kinematic flow models.

The multiresolution method has been devised (at least, originally) to reduce the computational cost of high resolution methods. In standard situations, the solution of a conservation law exhibits strong variations (shocks) in small regions but behaves smoothly on the major portion of the computational domain. The multiresolution technique adaptively concentrates computational effort on regions of strong variation. It goes back to Harten [17] for hyperbolic equations and was used by Bihari [7] and Roussel et al. [22] for parabolic equations.

Multiresolution methods for conservation

laws in several space dimensions are analyzed by Dahmen et al. [12], while fully adaptive multiresolution finite volume schemes, including an optimized adaptive memory storage, are presented by Cohen et al. [11]. See Chiavassa et al. [10] for a recent review on multiresolution methods for hyperbolic conservation laws.

Adaptive methods can be separated into two classes: one based on grid refinement to resolve gradients of a physically relevant quantity (see e.g. [19]), the other based on a posteriori error estimators, see e.g. [2] and the references therein. The present paper belongs to the first of these classes.

Interpolating wavelets [13, 18] are efficiently combined with linear or nonlinear thresholding strategies in order to produce sparse approximations on a near optimal grid. It is this family of wavelets that are used in this paper. The sparse point representation technique has been also applied to partial differential equations in [14], where an adaptive block refinement (ABR) method has been designed with the quadtree structure of dyadic blocks. Recently an adaptive multiresolution method with local time stepping has been developed in [15], nevertheless the method considered here still has an explicit time stepping.

We recall that the system (1) is called *hyperbolic* at a point (ϕ_1, ϕ_2) if the Jacobian \mathcal{J}_f of

the flux vector $\mathbf{f} = (f_1, f_2)^\top$,

$$\mathcal{J}_f = \begin{bmatrix} \frac{\partial f_1}{\partial \phi_1} & \frac{\partial f_1}{\partial \phi_2} \\ \frac{\partial f_2}{\partial \phi_1} & \frac{\partial f_2}{\partial \phi_2} \end{bmatrix} =: \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

has real eigenvalues, i.e., if the discriminant

$$\Delta(\phi_1, \phi_2) := ((J_{11} - J_{22})^2 + 4J_{12}J_{21})(\phi_1, \phi_2) \quad (2)$$

is positive. System (1) is *strictly hyperbolic*, if these eigenvalues are moreover distinct. If $\mathcal{J}_f(\phi_1, \phi_2)$ has a pair of complex conjugate eigenvalues (i.e., $\Delta(\phi_1, \phi_2) < 0$), then (1) is called *elliptic* at that point.

The initial conditions considered in this work are contained in the elliptic region, \mathcal{E}_1 , which is a subset of the phase space, where the eigenvalues of the Jacobian of the flux function are non-real. Two kinematic flow models for the initial value problems are considered:

Model 1: A prototype hyperbolic-elliptic system (Frid and Liu [16])

Frid and Liu [16] studied the following system of two scalar conservation laws in two unknowns for $\Phi = (\phi_1, \phi_2)^\top$,

$$\Phi_t + \mathbf{f}(\Phi)_x = 0, \quad (3)$$

where we define

$$\mathbf{f}(\phi_1, \phi_2) = \begin{pmatrix} \frac{1}{\sqrt{3}} \left(\frac{\phi_2^2}{2} - \frac{\phi_1^2}{2} + \phi_2 \right) + \phi_1 \\ \frac{1}{\sqrt{3}} (\phi_1 \phi_2 - \phi_1) + \phi_2 \end{pmatrix} \quad (4)$$

and consider the initial datum

$$\Phi(x, 0) = \Phi_0(x), \quad x \in \mathbb{R}. \quad (5)$$

This model does not refer to a specific application, but can serve as a prototype of, and has properties similar to, mixed systems of conservation laws modelling three-phase flow in porous media [16]. We here assume that $\Phi \in \nabla$, where

$$\nabla := \{(\phi_1, \phi_2) \in \mathbb{R}^2 \mid \phi_2 < 1, \phi_2 \pm \sqrt{3}\phi_1 \geq -2\} \quad (6)$$

Observe that $\mathbf{f}(\Phi) \in \nabla$ if $\Phi \in \nabla$. This implies that the solution $\Phi(\cdot, t)$ of the system will take its values in ∇ if $\Phi_0(x) \in \nabla$ for all $x \in \mathbb{R}$, [16].

The eigenvalues of the Jacobian \mathcal{J}_f are given by

$$\lambda_{1,2} = 1 \mp \frac{1}{\sqrt{3}} \sqrt{\phi_1^2 + \phi_2^2 - 1}, \quad (7)$$

such that the system (4) is of mixed type, being hyperbolic for $\|\Phi\|_2^2 = \phi_1^2 + \phi_2^2 > 1$ and elliptic for

$$\Phi \in \mathcal{E}_1 := \{\Phi = (\phi_1, \phi_2) \in \nabla \mid \phi_1^2 + \phi_2^2 < 1\}.$$

The elliptic region \mathcal{E}_1 , the unit circle, is tangent to each of the three sides of ∇ . The absolute values of the eigenvalues are $|\lambda_1| = |\lambda_2| = 1$ on the boundary of the elliptic region and increase toward the origin, where $|\lambda_1| = |\lambda_2| = 2\sqrt{3}/3$.

Model 2: Settling of a heavy-buoyant bidisperse suspensions

Model 2 emerges from a general theory of sedimentation of polydisperse suspensions with N particle species, which is presented in detail in [4] and is briefly outlined here, considering $N = 2$. The suspension is composed of rigid spherical particles which are dispersed in a viscous fluid of density ϱ_f and viscosity μ_f . The particles belong to N different species having size (diameter) d_i and density ϱ_i , $i = 1, \dots, N$, where $d_i \neq d_j$ or $\varrho_i \neq \varrho_j$ for $i \neq j$, and $d_1 \geq d_2 \geq \dots \geq d_N$.

Model equations for the three-dimensional motion of the mixture were derived in [5] from the mass and linear momentum balances for the fluid and each solid species, introducing constitutive assumptions and simplifying the model equations as a consequence of a dimensional analysis. The relevant parameters are $\delta_i := d_i^2/d_1^2$ and $\bar{\varrho}_i := \varrho_i - \varrho_f$ for $i = 1, \dots, N$. Here, ϕ_{\max} denotes a maximum solids volume fraction, which we here assume to be constant, so that the relevant phase space becomes

$$\mathcal{D}_{\phi_{\max}} = \left\{ \Phi \in \mathbb{R}^N \mid \phi_1 \geq 0, \dots, \phi_N \geq 0, \right. \\ \left. , \phi_1 + \dots + \phi_N \leq \phi_{\max} \right\}.$$

Moreover, we introduce the vector $\bar{\varrho} := (\bar{\varrho}_1, \dots, \bar{\varrho}_N)^\top$, the cumulative solids fraction $\phi := \phi_1 + \dots + \phi_N$, the viscosity parameter $\mu := -gd_1^2/(18\mu_f) < 0$, where g is the acceleration of gravity, and the hindered settling factor $V = V(\phi)$, which may be chosen as

$$V(\phi) = \begin{cases} (1 - \phi)^{n-2} & \text{if } \Phi \in \mathcal{D}_{\phi_{\max}} \\ 0 & \text{otherwise,} \end{cases} \quad n > 2. \quad (8)$$

Then the phase velocity of particle species i is given by

$$\begin{aligned} v_i(\Phi) &= \mu V(\phi) [\delta_i(\bar{\rho}_i - \bar{\rho}^T \Phi) - B], \\ B &= \sum_{m=1}^N \delta_m \phi_m (\bar{\rho}_m - \bar{\rho}^T \Phi), \quad i = 1, \dots, N. \end{aligned} \quad (9)$$

For one-dimensional batch settling of a suspension with initially given composition in a closed vessel of depth L , the governing equation is

$$\begin{aligned} \partial_t \Phi + \partial_x \mathbf{f}(\Phi) &= 0, \quad x \in (0, L), \quad t > 0, \\ \mathbf{f}(\Phi) &= (f_1(\Phi), \dots, f_N(\Phi))^T, \\ f_i(\Phi) &= \phi_i v_i(\Phi), \quad i = 1, \dots, N, \end{aligned} \quad (10)$$

where $v_i(\Phi)$ is given by (9), and we consider the initial and zero-flux boundary conditions

$$\Phi(x, 0) = \Phi_0(x), \quad x \in [0, L], \quad (11)$$

$$\mathbf{f}|_{x=0} = \mathbf{f}|_{x=L} = 0. \quad (12)$$

For this model, the criterion for ellipticity is equivalent to the instability criterion by Batchelor and Janse van Rensburg [1]. In [5] it is shown that loss of hyperbolicity, that is the occurrence of complex eigenvalues of $\mathcal{J}_{\mathbf{f}}(\Phi)$, provides an instability criterion for polydisperse suspensions for arbitrary N . For $N = 3$, this criterion can be evaluated by a convenient calculation of a discriminant, which is an explicit algebraic function of pointwise values of the partial derivatives $\partial f_i / \partial \phi_j$. Biesheuvel et al. [6] and Bürger et al. [5] determine instability regions for $N = 2, 3$ and different choices of $\mathbf{f}(\Phi)$. Berres et al. [4] proved that for equal-density particles ($\bar{\rho}_1 = \dots = \bar{\rho}_N = \rho_s - \rho_f$) and arbitrary particle size distributions with $\delta_i \neq \delta_j$ for $i \neq j$, the system (10) with the flux vector $\mathbf{f}(\Phi)$ defined via (9) is strictly hyperbolic for all $\Phi \in \mathcal{D}_1$ with $\phi_1 > 0, \dots, \phi_N > 0$ and $\phi < 1$. The instability criterion for one-dimensional batch settling is the same as for the full two- or three-dimensional model, in which the corresponding first-order system of conservation laws is coupled with additional equations of motion for the mixture.

Biesheuvel et al. [6] provide a vivid description of the consequences of lack of stability, which include the formation of blobs and “fingers” in bidisperse sedimentation, increased sedimentation rates, decreased separation quality of hydraulic classifiers, and non-homogeneous sediments in material manufacturing by

suspension processing. These phenomena have indeed been observed in experiments under the circumstances predicted by the instability criterion. On the other hand, the hyperbolicity, and thus stability result for equal-density spheres agrees with experimental evidence, since instabilities never have been observed with this type of mixtures, but always involve particles of different densities.

Numerical simulations are presented for one-dimensional kinematic flow models, related to the problem of sedimentation of polydisperse suspensions with 2 particles species [4].

The lack of stability may lead to anomalous behavior of the numerical solution, for example to oscillations or a “locking” effect, i.e., heavy and buoyant particles block each other within the vessel, as presented in [3].

The numerical results indicate that the emergence of wild oscillations associated to the elliptic degeneracy is related to the location and shape of the elliptic region with respect to the invariance region. Wild oscillations only appear when the elliptic region is tangent to each side of the invariance region.

Homogenous initial data which are perturbed in the elliptic region show some spiraling: in the phase space there appears a spiral starting at the homogeneous data.

The goal of the numerical experiments of this work is to revisit the Frid-Liu and Model 2 examples and try to find out where the oscillations come from. In [16] the numerical example takes constant initial values which are perturbed at a small number of grid points in the vicinity of zero. In both cases considered, the initial conditions are taken inside \mathcal{E}_1 , in one ($\Phi_0 \equiv (-0.9, 0.1)$) oscillations can be observed, whereas in the other ($\Phi_0 \equiv (0.1, 0.2)$) not, but only peaks propagate away from the perturbation with an amplitude decaying due to numerical viscosity.

To come closer to the anatomy of oscillations, now we are going to consider a Riemann problem having the left state inside \mathcal{E}_1 , and the right state outside.

The oscillation presents a mixed oscillatory pattern, as can be observed on Figure 1(a) and (b): Towards the flow direction, there are strong oscillations followed by a smooth part of the solution. This mixed structure with large smooth parts makes clear the advantage of an

adaptive multiresolution scheme, which can be recoarsened in the smooth parts.

A zoom into the oscillations shows that there is no convergence in a traditional sense but the oscillation frequency depends on the refinement level, as is illustrated on Figure 2(d)

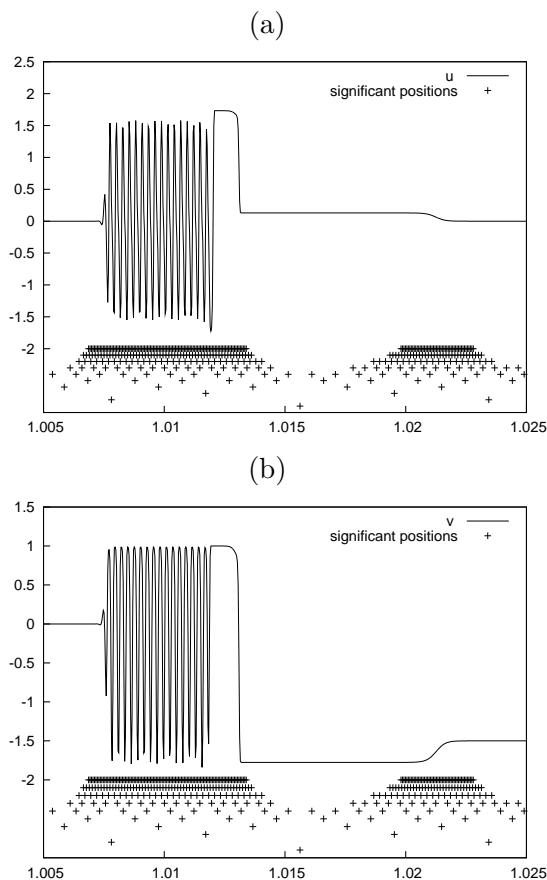


Figure 1: Frid-Liu example, Riemann problem with $\Phi_l = (0, 0)$, $\Phi_r = (0, -1.5)$. (a), (b): Profiles of ϕ_1 and ϕ_2 . The final simulation time is 0.0153s, keeping the CFL equal 0.125 as done in [16].

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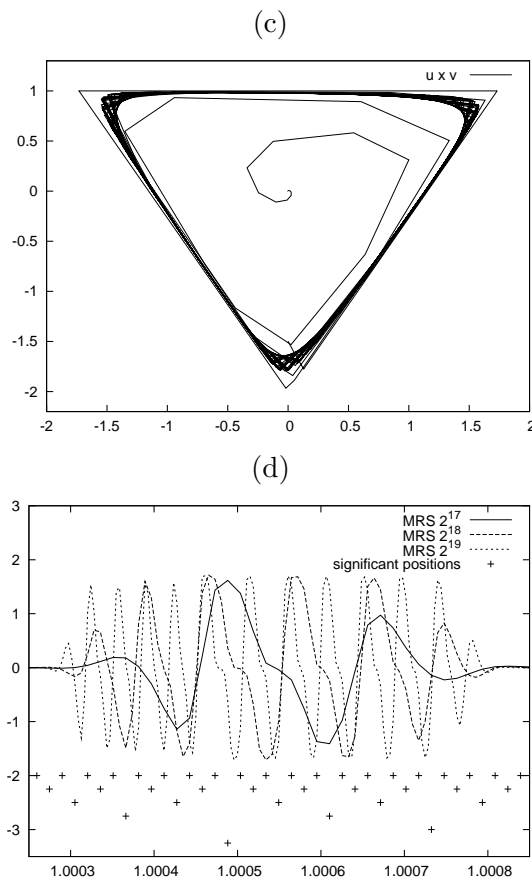


Figure 2: Frid-Liu example, Riemann problem with $\Phi_l = (0, 0)$, $\Phi_r = (0, -1.5)$. (c) Solution at the same time step in the $\phi_1 \times \phi_2$ plane, (d) Zoom of profile into oscillations with refinements considering the finest resolution level of the multiresolution with 2^{17} , 2^{18} and 2^{19} .

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