Note on Lie point symmetries of Burgers Equations'

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Abstract: *In this work we study the Lie point symmetries of a class of evolution equations and obtain a group classification of these equations. We also identify the classical Lie algebras that the symmetry Lie algebras are isomorphic to.*

Key words: *Evolution equations, Lie point symmetry, symmetry Lie algebras*

1 Introduction

Let $x \in M \subseteq \mathbb{R}^n$, M open, $u : M \to \mathbb{R}$ a smooth function and $k \in \mathbb{N}$. We use $\partial^k u$ to denote the jet bundle corresponding to all kth partial derivatives of u with respect to x. We simply denote $\partial^1 u$ by ∂u.

A Lie point symmetry[1] of a PDE $F = F(x, u, \partial u, \cdots, \partial^m u) = 0$ of order m is a vector field

$$S = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u}$$

on $M \times \mathbb{R}$ such that $S^{(m)}F = 0$ when $F = 0$ and

$$S^{(m)} := S + \eta_i^{(1)}(x,u,\partial u)\frac{\partial}{\partial u_i} + \cdots + \eta^{(m)}_{i_1\cdots i_m}(x,u,\partial u,\cdots,\partial^m u)\frac{\partial}{\partial u_{i_1\cdots i_m}} \tag{1}$$

is the extended symmetry on the jet space $(x, u, \partial u, \cdots, \partial^m u)$. Here the summation over the repetead indeces is understood. For the definition of functions $\eta_i^{(1)}, \cdots, \eta^{(j)}_{i_1\cdots i_j}$ in (1), see [2], p. 67.

Some applications of this method in differential equations can be found in [2, 3, 4, 5, 8, 9, 10, 11, 12, 15, 16]. We shall not present more preliminaries concerning the Lie point symmetries of differential equations supposing that the reader is familiar with the basic notions and methods of group analysis [2, 8, 11].

In a previous paper, Lahno and Samoilenko [9] studied the group classification of quasilinear equation

$$u_t = F(x,t,u,u_x)u_{xx} + G(x,t,u,u_x), \tag{2}$$

where $u = u(x,t)$, for general smooth functions F and G.

When $F = 1$ and $G = -u\,u_x$, the equation is communly known as Burgers' equation because it was first studied by Burgers in the last century. In this letter we shall call Burgers' equation the general case of equation (2) when $F = \nu = const.$ and $G = -g(u)u_x$, where $g(u)$ is a smooth function.

[1] In fact, a Lie point symmetry is given by the exponential map $(\exp S)(x,u) =: (x^*, u^*) \in \mathbb{R}^n \times \mathbb{R}$. We are identifying the point transformation with its generator.

In [12], the group classification of (2) was carried out with $F = 0$ and $G = -g(u)u_x$, for particular choices of the function g. In [10], the results obained in [12] were generalized. For more details, see [12, 10].

The purpose of the present note is to illuminate the properties of the Burgers' equation

$$\nu u_{xx} = u_t + g(u)u_x \tag{3}$$

from the point of view of the S. Lie Symmetry Theory. We are interested in bringing together some results from literature and add some new results concerning the group classification of Eq. (3).

The linear case $g(u) = k = const.$ shall not be considered here because we are interested only in nonlinear cases. The Lie point symmetries of the particular case $g(u) = 0$ can be found in [2, 11].

For the Burgers' equation

$$u_{xx} = u_t + uu_x,$$

see the Lie point symmetries in [2, 11].

For a discussion of group classification of diffusion-convection equations and applications of Lie point symmetry theory in evolution equations, see [15] and references therein.

The importance of group classification of differential equations was first emphasized by Ovsiannikov in 1950s-1960s, when he and his school began a systematic research program of successfully applying modern group analysis methods to wide range of physically important problems.

Following Olver ([11], p. 182), we recall that to perform a group classification on a differential equation involving a generic function g consists of finding the Lie point symmetries of the given equation with arbitrary g, and, then, to determine all possible particular forms of g for which the symmetry group can be enlarged.

It is worth observing that for problems which arise from physics, quite often there exists a physical motivation for considering such specific cases. For example, see the group classifications presented in [3, 4, 10, 12, 15, 16] and references therein.

The remaining of the paper is organized as follows. In section 2 we carry out the complete group classification of Burgers' equation and in the section 3 we identify the classical Lie algebras that the symmetry Lie algebras are isomorphic to.

2 Main result

Let us consider the equation (3) with $\nu \neq 0$ and $g'(u) \neq 0$. In the remaining of this paper, we shall suppose that all functions are smooths and they are well defined.

Lemma 1. *Let*

$$S = \xi(x,t,u)\frac{\partial}{\partial x} + \phi(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \tag{4}$$

be a symmetry of equation (3)*. Then* $\xi = \xi(x,t)$*,* $\phi = \phi(t)$ *and* $\eta = \alpha(x,t)u + \beta(x,t)$*.*

Proof. From [1, 2], we conclude that $\xi = \xi(x,t)$, $\phi = \phi(x,t)$ and $\eta = \alpha(x,t)u + \beta(x,t)$. From [9], $\phi = \phi(t)$. □

Lemma 2. *The linearly independet set of determining equations of equation* (3) *is:*

$$\phi'(t) = 2\xi_x, \tag{5}$$

$$u\alpha_t - \nu u\alpha_{xx} + ug(u)\alpha_x + \beta_t - \nu\beta_{xx} + g(u)\beta_x = 0, \tag{6}$$

$$\xi_t + 2\nu\alpha_x - g(u)\xi_x - ug'(u)\alpha - g'(u)\beta = 0. \tag{7}$$

Proof. It follows from the invariance condition $S^{(2)}F = 0$ whenever

$$F = \nu u_{xx} - u_t - g(u)u_x = 0.$$

See also [9, 6, 7, 15]. □

Remark 1:

1. The number of determining equations is greater than those presented in Lemma 2, but some of them are equivalent to others. Thus, we have listed only the linearly independent set of equations in order to determine the symmetries.

2. We have used especific Mathematica packages in order to obtain the determining equations and solve them. See [6, 7].

Theorem 1. *The widest Lie point symmetry group of Burgers' equation* (3) *with an arbitrary* $g(u)$, *is determined by the operators*

$$X = \frac{\partial}{\partial x}, \quad T = \frac{\partial}{\partial t}. \tag{8}$$

For some special choices of the function $g(u)$ *it can be extended in the cases listed below. We shall write only the generators additional to* (8).

1. *If* $g(u) = u$, *then* $B_{11} = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (x - tu)\frac{\partial}{\partial u}$, $B_{12} = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$ *and* $B_{13} = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$.
2. *If* $g(u) = u^p$, $p \neq 0,\ 1$, *then the additional generator is* $B_2 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \frac{1}{p}u\frac{\partial}{\partial u}$.
3. *If* $g(u) = \log u$, *then the additional generator is* $B_3 = t\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$.
4. *If* $g(u) = e^{bu}$, $b = const. \neq 0$, *then* $B_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \frac{1}{b}\frac{\partial}{\partial u}$.
5. *If* $g(u) = \frac{1-u}{1+u}$, *then* $B_5 = (x - t)\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + (1 + u)\frac{\partial}{\partial u}$.
6. *If* $g(u) = \frac{1}{1+u}$, *then* $B_6 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + (1 + u)\frac{\partial}{\partial u}$.
7. *If* $g(u) = \frac{u}{1+u}$, *then the additional generator is* $B_7 = (x + t)\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + (1 + u)\frac{\partial}{\partial u}$.
8. *If* $g(u) = \frac{u}{1-u}$, *then the additional generator is* $B_8 = (x - t)\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + (u - 1)\frac{\partial}{\partial u}$.
9. *If* $g(u) = \frac{1+u}{u}$, *then the additional generator is* $B_9 = (x + t)\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$.

Proof. If g is an arbitrary function, in order that the equations (6) and (7) be true, we necessarily have $\xi_t + 2\nu\alpha_x = \xi_x = \alpha = \beta = 0$. Then, from equations (5) and (7) we conclude that $\xi = c_1 = const.$ and $\phi = c_2 = const.$. Thus, the symmetry (4) is spanned by translations in x and t.

The proof of the case $g(u) = u$ can be found in [2, 11]. For the other cases, substituting the functions listed in the Theorem in equations (6) and (7), we obtain two identities in terms of u, $g(u)$, $ug(u)$, $g'(u)$ and $ug'(u)$. After solving it (or using [6, 7]), we obtain the coefficients ξ, ϕ, α and β of symmetry (4). □

Remark 2:

1. The arbitrary case is easy to understand. Since Eq. (3) has no coefficients depending on x and t, the translational invariance is immediate.

2. We emphasize that the cases $g(u) = u^p$ (including $p = 1$), $g(u) = e^{bu}$ (with $b \neq 0$) and $g(u) = \log u$ are known. See [15], for example. They are listed in Theorem 1 for completeness.

3. To the best of our knowledge, the additional symmetries of cases 5 to 9 are not listed previously in any paper.

3 Symmetry Lie algebras

In this section we are interested in classifing the symmetry Lie algebras of equation (3). In the next theorem we present only the non-null Lie brackets.

Theorem 2. *The symmetry Lie algebras of the Burgers' equations are as follows.*

1. *If* $g(u) = u$, *then* $[X, B_{11}] = B_{12},\ \ [X, B_{13}] = X,\ \ [T, B_{11}] = B_{13}, [T, B_{12}] = X,$

   $[X, B_{13}] = 2T,\ \ [B_{11}, B_{13}] = -2B_{11},\ \ [B_{12}, B_{13}] = -B_{12}.$
2. *If* $g(u) = u^p,\ p \neq 0,\ 1,$ *then* $[X, B_2] = X,\ \ [T, B_2] = 2T.$
3. *If* $g(u) = \log u,$ *then* $[T, B_3] = X.$
4. *If* $g(u) = e^{bu},\ b = const.,$ *then* $[X, B_4] = X,\ \ [T, B_4] = 2T.$
5. *If* $g(u) = \dfrac{1-u}{1+u},$ *then* $[X, B_5] = X,\ \ [T, B_5] = -X + 2T.$
6. *If* $g(u) = \dfrac{1}{1+u},$ *then* $[X, B_6] = X,\ \ [T, B_6] = 2T.$
7. *If* $g(u) = \dfrac{u}{1+u},$ *then* $[X, B_7] = X,\ \ [T, B_7] = X + 2T.$
8. *If* $g(u) = \dfrac{u}{1-u},$ *then* $[X, B_8] = X,\ \ [T, B_7] = -X + 2T.$
9. *If* $g(u) = \dfrac{1+u}{u},$ *then* $[X, B_9] = X,\ \ [T, B_7] = X + 2T.$

Let $\mathfrak{g}_1 := \{X, T, B_{11}, B_{12}, B_{13}\}$ and $\mathfrak{g}_i := \{X, T, B_i\}$, $2 \leq i \leq 9$.

It is immediate that $\mathfrak{g}_2 \cong \mathfrak{g}_4 \cong \mathfrak{g}_6$, $\mathfrak{g}_5 \cong \mathfrak{g}_8$, $\mathfrak{g}_7 \cong \mathfrak{g}_9$ and that, under the change $X \mapsto -X$, $\mathfrak{g}_5 \cong \mathfrak{g}_7$.

Theorem 3. $\mathfrak{g}_2 \cong \mathfrak{g}_7$.

Proof. Let $e_1 := X$, $e_2 := X + T$ and $e_3 := B_7$. Then,

$$[e_1, e_3] = e_1 \text{ and } [e_2, e_3] = 2e_2. \tag{9}$$

□

The following result is a consequence from Theorems 2, 3, [13, 14] and the change $e'_1 = e_2$, $e'_2 = e_1$, $e'_3 = \frac{1}{2}e_3$ in (9).

Theorem 4. $\mathfrak{g}_1 \cong A_{5,40}$, $\mathfrak{g}_2 \cong \mathfrak{g}_4 \cong \mathfrak{g}_5 \cong \mathfrak{g}_6 \cong \mathfrak{g}_7 \cong \mathfrak{g}_8 \cong \mathfrak{g}_9 \cong A^{\frac{1}{2}}_{3,5}$, $\mathfrak{g}_3 \cong A_{3,1}$, *where* $A_{3,1}$ *is the Weyl-Heisenberg algebra.*

4 Conclusions

In this paper, the group classification of Burgers' equation (Eq. (3)) is performed. The main result is Theorem 1, where is presented the group classification of Eq. (3). Theorem 4 show us the classical Lie algebras that the symmetry Lie algebras are isomorphic to.

Some of the cases presented in Theorem 1 (1–4) are known. However, we do not aware of any paper containing the other cases.

The most interesting fact concerning to the symmetry Lie algebras is, except to the classical Burgers equation and the *log* nonlinearity, all other cases have symmetry Lie algebras isomorphic.

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