Curvature (k_2, k_3) formulas for implicit curves in n-dimensions

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Abstract: We derive curvatures (k_2, k_3) for transversal intersections of intersection curves of $(n-1)$ implicit surfaces in $\mathbb{R}^n$.
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1 Introduction

The purpose of this work is provide cuvatures (k_2, k_3) formula for implicit curves in (n)-dimension, that is, to curves generated by the intersection of $(n-1)$ implicit equation: $F_1(x_1, \ldots, x_n) = 0 \cap \ldots \cap F_{(n-1)}(x_1, \ldots, x_n) = 0$. Initially, the motivation of this paper was an open problem, proposed by R. Goldman. In his paper, he proposed the following open problem: **Problem 1:** Derive closed formulas for higher curvatures for implicit curves in $(n+1)$-dimensions. While differential geometry of a parametric curve in $\mathbb{R}^3$ can be found in textbooks such as in (Struik, 1950; Wilmore, 1959; Stoker, 1969; Spivak, 1975; do Carmo, 1976), differential geometry of a parametric curve in $\mathbb{R}^n$ can be found in textbook such as in (Klingenberg, 1978), there is little literature on differential geometry of intersection curves in $\mathbb{R}^3$ and, rarely, in $\mathbb{R}^4$ and $\mathbb{R}^n$. Willmore (1959) describes how to obtain the curvature k and the torsion τ of the intersection curve of two implicit surfaces in $\mathbb{R}^3$. Hartmann (1996) provides formulas for computing the curvature k of the intersection curves for all three types of intersection problems in $\mathbb{R}^3$, using the implicit function theorem. Ye and Maekawa (1999) provides k, τ, using the vector α'' as linear combination of the normal vectors of the surfaces and α''' as linear combination of the tangent vector and normal vectors of the surfaces. Goldman (2005) provides formulas for computing the curvature k and torsion τ of intersection curve of two implicit surfaces in $\mathbb{R}^3$ and curvature one (k_1) of intersection curve in (n)-dimensions. Aléssio in (2006) provides formulas for computing the curvature k and torsion τ of the intersection curves of two implicit surface in $\mathbb{R}^3$, using the implicit function theorem. Aléssio in (2009) provides formulas for computing the curvature k_1, k_2, k_3 of the intersection curves of two implicit surface in $\mathbb{R}^4$, using the implicit function theorem and generalizing the method of X. Ye and T. Maekawa for 4-dimension. In this work, we derive curvatures k_2 and k_3 for implicit curves in (n)-dimensions, operating T^* with ∇ two $\nabla(\nabla T^*)$ and three times $\nabla(\nabla(\nabla T^*))$, for obtain $\left(\dddot{\beta}, \ddddot{\beta}\right)$ and using outer product for obtain (k_2, k_3) .

2 Review the of Differential Geometry and Outer Product

Where $\wedge$ is outer product of the **three** and **four** vectors in (n) $-$dimensional, $\times$ is cross product of the $(\mathbf{n-1})$ vectors in (n) $-$dimensional and $*$ product between matrices.

The extension of the c**ross product** $(\times)$ to n-dimensions that generates a vector perpendicular to a collection of $n-1$ vectors is given by a determinant. Let $e = (e_1, e_2, \cdots, e_n)$ be the canonical basis for $\mathbb{R}^n$. Then

$$\nabla F_1 \times \ldots \times \nabla F_{n-1} = Det \begin{pmatrix} e \\ \nabla F_1 \\ \vdots \\ \nabla F_{n-1} \end{pmatrix} = Det \begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial F_{n-1}}{\partial x_1} & \frac{\partial F_{n-1}}{\partial x_2} & \cdots & \frac{\partial F_{n-1}}{\partial x_n} \end{pmatrix}$$

The **outer product** ($\wedge$) of the r vectors in $n-$dimensional

Let $(e_1, ..., e_n)$ be an orthonormal basis. If $u_1 = \sum_i \alpha_1^i e_i$, ..., $u_r = \sum_i \alpha_r^i e_i$, the outer product of r-vectors is $u_1 \wedge ... \wedge u_r = \sum_J \det(\alpha^J) e_J$ where $e_j = e_{j_1} \wedge ... \wedge e_{j_r}$, $J = \{j_1, ..., j_r\}$, $1 \leq j_1 \leq \cdots \leq j_r \leq (n)$. Then $\alpha^J = \begin{vmatrix} \alpha_1^{j_1} & \alpha_2^{j_1} & \dots & \alpha_r^{j_1} \\ \alpha_1^{j_2} & \alpha_2^{j_2} & \dots & \alpha_r^{j_2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{j_r} & \alpha_2^{j_r} & \dots & \alpha_r^{j_r} \end{vmatrix}$ is the matrix $r \times r$.

2.0.1 The Local Theory of Curves Parametrized Arbitrary.

Definition 1 *Let $\beta : (a, b) \to \mathbb{R}^n$ be a regular curve, and let $\alpha : (c, d) \to \mathbb{R}^n$ be a unit-speed parametrization of β. Write $\beta(t) = \alpha(s(t))$ (where $s(t)$ is just the arc length function). Denote by k_i, $i = 1, 2, 3, ..., n$ the curvatures. Also, let $\{\mathbf{b}_1, ..., b_n\}$ be the Frenet frame field of α and $\{\overline{\mathbf{b}}_1, ..., \overline{\mathbf{b}}_n\}$ be the Frenet frame field of β. Then we define*

$\overline{k}_i(t) = k_i(s(t)), i = 1, ..., n$ *and* $\overline{b}_i(t) = b_i(s(t)), i = 1, ..., n.$

Definition 2 *Let $\beta : (a, b) \to \mathbb{R}^n$ be a regular curve whit speed $v(t) = \left\| \dot{\beta}(t) \right\| = \dot{s}(t)$ and let $\alpha : (c, d) \to \mathbb{R}^n$ be a unit-speed parametrization of β. Write $\beta(t) = \alpha(s(t))$ where $s(t)$ is just the arc length function. Then, the Frenet formulas of β are:*

$$\frac{d}{dt}\left(\overline{\mathbf{b}}_i(t)\right) = -v(t)\overline{\kappa}_{(i-1)}(t)\overline{\mathbf{b}}_{i-1}(t) + v(t)\overline{\kappa}_i(t)\overline{\mathbf{b}}_{i+1}(t),$$

where $\overline{\kappa}_0 = \overline{\kappa}_n = 0$, $\overline{\mathbf{b}}_0 = \overline{\mathbf{b}}_{n+1} = \mathbf{0}$.

The following properties characterize some special curves:

$\overline{\kappa}_i(t) = 0, i \in \{1, \cdots, n-1\}$ if and only if β is lying in a $i-$dimensional subspace of $\mathbb{R}^{i+1}$.

Lemma 1 *The vectors $\dot{\beta}$, $\ddot{\beta}$, $\dddot{\beta}$, and $\ddddot{\beta}$ of a regular curve β are given by*

$$\dot{\beta}(t) = v\overline{\mathbf{b}}_1(t) \tag{1}$$

$$\ddot{\beta}(t) = \frac{dv}{dt}\overline{\mathbf{b}}_1(t) + v^2\overline{k}_1(t)\overline{\mathbf{b}}_2(t) \tag{2}$$

$$\dddot{\beta}(t) = \left(\frac{d^2v}{dt^2} - v_1^3\overline{k}_1^2(t)\right)\overline{\mathbf{b}}_1(t) + \left(3v\frac{dv}{dt}\overline{k}_1(t) + v^2\frac{d\overline{k}_1(t)}{dt}\right)\overline{\mathbf{b}}_2(t) + v^3\overline{k}_1(t)\overline{k}_2(t)\overline{\mathbf{b}}_3(t) \tag{3}$$

$$\ddddot{\beta}(t) = a_{14}(t)\overline{\mathbf{b}}_1(t) + a_{24}(t)\overline{\mathbf{b}}_2(t) + a_{34}(t)\overline{\mathbf{b}}_3(t) + v^4\overline{k}_1(t)\overline{k}_2(t)\overline{k}_3(t)\overline{\mathbf{b}}_4(t), \tag{4}$$

where v denotes the speed of β.

2.1 Implicit Surface

Definition 3 (Implicit Surface) *Let $f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable mapping of an open set D. Given $c \in \mathbb{R}$, we remember that the level set c of the f is the set defined as*
$f^{-1}(c) = \{(x_1, ..., x_n) \in D; f(x_1, ..., x_n) = c\}$, i.e., $f^{-1}(c)$ is the set of the solutions in D of the equations $f(x_1, ..., x_n) = c$.

Proposition 1 ***[Regular Surfaces]*** *If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a differentiable function and $c \in f(U)$ is a regular value of f, then $f^{-1}(c)$ is a regular surface in $\mathbb{R}^n$. The implicit surface f is regular if $\nabla f = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right) \neq 0$.*

Definition 4 ***Transversal Intersection Curve*** *Three regular surfaces $S_1 = F_1^{-1}(c)$, $S_2 = F_2^{-1}(c)$, ..., $S_{(n-1)} = F_{n-1}^{-1}(c)$ intersect each other transversally if, whenever $p \in S_1 \cap S_2 \cap ... \cap S_{n-1}$, then $N^{F_1}(p)$, $N^{F_2}(p)$, ..., $N^{F_{n-1}}(p)$ are not parallel, i.e., linearly independent.*

3 Curves in Surface

Consider an implicit surface represented by $F_1, F_2, ..., F_{n-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$. A curve $x_1 = x_1(t)$, $x_2 = x_2(t)$, ..., $x_n = x_n(t)$, in the n-dimensional space, defines a implicit curve $\beta(t) = \{(x_1(t), ..., x_n(t)) \mid F_1(\beta(t)) = 0 \cap ... \cap F_{n-1}(\beta(t)) = 0\}$ on (n)-dimesional space. Then, we have

$$\dot{\beta}(t) = \nabla F_1 \times ... \times \nabla F_{n-1}, \tag{5}$$

$$\ddot{\beta}(t) = (\nabla F_1 \times ... \times \nabla F_{n-1}) * \nabla(\nabla F_1 \times ... \times \nabla F_{n-1}), \tag{6}$$

$$\dddot{\beta}(t) = T^* * \nabla(T^*) * \nabla(T^*) + T^* * \nabla(\nabla(T^*)) * (T^*)^T, \tag{7}$$

$$\ddddot{\beta}(t) = \dddot{\beta}(t) * \nabla(T^*) + 3\ddot{\beta}(t) * \nabla(\nabla(T^*)) * (T^*)^T + (T^*) * [T^* * \nabla(\nabla(\nabla(T^*)))] * (T^*)^T \tag{8}$$

where $\dot{\beta}^T(x) = \begin{bmatrix} \dot{x}_1 & \dot{x}_2 & \cdots & \dot{x}_n \end{bmatrix}, \cdots, \ddddot{\beta}^T(x) = \begin{bmatrix} \ddddot{x}_1 & \ddddot{x}_2 & \cdots & \ddddot{x}_n \end{bmatrix}$,
$\nabla f = \begin{bmatrix} f_{x_1} & f_{x_2} & \cdots & f_{x_n} \end{bmatrix}$, $\quad T^* = \nabla F_1 \times ... \times \nabla F_{n-1} = (T^*_{11}, T^*_{12}, \cdots, T^*_{1n})$.

Here $\nabla(T^*)$ means apply ∇ to each element T^*_{1k} of the vector T^* to generate a column, $\nabla^T(T^*_{1k}) = \begin{bmatrix} (T^*_{1k})_{x_1} \\ (T^*_{1k})_{x_2} \\ \vdots \\ (T^*_{1k})_{x_n} \end{bmatrix}$, then $\nabla(T^*) = \begin{bmatrix} (T^*_{11})_{x_1} & \cdots & (T^*_{1k})_{x_1} & \cdots & (T_{1n})_{x_1} \\ (T^*_{11})_{x_2} & \cdots & (T^*_{1k})_{x_2} & \cdots & (T^*_{1n})_{x_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (T^*_{11})_{x_n} & \cdots & (T^*_{1k})_{x_n} & \cdots & (T^*_{1n})_{x_n} \end{bmatrix}$,
and $\nabla(\nabla(T^*))$ means apply ∇ to each column vector of the matrix $\nabla(T^*)$ to generate a matrix,

$$H_{1k} = \nabla(\nabla^T(T^*_{1k})) = \nabla \begin{bmatrix} (T^*_{1k})_{x_1} \\ (T^*_{1k})_{x_2} \\ \vdots \\ (T^*_{1k})_{x_n} \end{bmatrix} = \begin{bmatrix} \nabla\left((T^*_{1k})_{x_1}\right) \\ \nabla\left((T^*_{1k})_{x_2}\right) \\ \vdots \\ \nabla\left((T^*_{1k})_{x_n}\right) \end{bmatrix} = \begin{bmatrix} (T^*_{1k})_{x_1x_1} & (T^*_{1k})_{x_1x_2} & \cdots & (T^*_{1k})_{x_1x_n} \\ (T^*_{1k})_{x_2x_1} & (T^*_{1k})_{x_2x_2} & \cdots & (T^*_{1k})_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ (T^*_{1k})_{x_nx_1} & (T^*_{1k})_{x_nx_2} & \cdots & (T^*_{1k})_{x_nx_n} \end{bmatrix},$$

then $\nabla(\nabla(T^*)) = \begin{bmatrix} H_{11} & \cdots & H_{1k} & \cdots & H_{1n} \end{bmatrix}$, is $1 \times n$ matrices,
and $\nabla(\nabla(\nabla(T^*)))$ means apply ∇ to each element H_{1k} (matrix) of the matrix $\nabla(\nabla(T^*))$, to generate a column $J_{1k} = \nabla^T(H_{1k}) = \begin{bmatrix} (H_{1k})_{x_1} \\ (H_{1k})_{x_2} \\ \vdots \\ (H_{1k})_{x_n} \end{bmatrix}$, of $n \times 1$ matrices, then

$$\nabla(\nabla(\nabla(T^*))) = \begin{bmatrix} J_{11} & \cdots & J_{1k} & \cdots & J_{1n} \end{bmatrix} = \begin{bmatrix} (H_{11})_{x_1} & \cdots & (H_{1k})_{x_1} & \cdots & (H_{1n})_{x_1} \\ (H_{11})_{x_2} & \cdots & (H_{1k})_{x_2} & \cdots & (H_{1n})_{x_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (H_{11})_{x_n} & \cdots & (H_{1k})_{x_n} & \cdots & (H_{1n})_{x_n} \end{bmatrix}$$, is $n \times n$ matrices.

Proof. To be simpler the calculations, we go to make for the case of the curve $\beta : I \to \mathbb{R}^3$. The Eq. (6) is a consequence

$$\ddot{\beta}(t) = \frac{d}{dt}\left(\dot{\beta}(t)\right)$$
$$\ddot{\beta}(t) = \frac{d}{dt}(\nabla F_1 \times \nabla F_2)$$
$$\ddot{\beta}(t) = \frac{d}{dt}(\nabla F_1) \times \nabla F_2 + \nabla F_1 \times \frac{d}{dt}(\nabla F_2)$$
$$\ddot{\beta}(t) = \left((\alpha'(s))^T HF_1\right) \times \nabla F_2 + \nabla F_1 \times \left((\alpha'(s))^T HF_2\right)$$
$$\ddot{\beta}(t) = \begin{bmatrix} T^*_{11} \\ T^*_{12} \\ T^*_{13} \end{bmatrix}^T \begin{bmatrix} F^1_{x_2x_1}F^2_{x_3} - F^1_{x_3x_1}F^2_{x_2} & \cdots & F^1_{x_1x_1}F^2_{x_2} - F^1_{x_2x_1}F^2_{x_1} \\ F^1_{x_2x_2}F^2_{x_3} - F^1_{x_3x_2}F^2_{x_2} & \cdots & F^1_{x_1x_2}F^2_{x_2} - F^1_{x_2x_2}F^2_{x_1} \\ F^1_{x_2x_3}F^2_{x_3} - F^1_{x_3x_3}F^2_{x_2} & \cdots & F^1_{x_1x_3}F^2_{x_2} - F^1_{x_2x_3}F^2_{x_1} \end{bmatrix} +$$
$$\begin{bmatrix} T^*_{11} \\ T^*_{12} \\ T^*_{13} \end{bmatrix}^T \begin{bmatrix} F^1_{x_2}F^2_{x_3x_1} - F^1_{x_3}F^2_{x_2x_1} & \cdots & F^1_{x_1}F^2_{x_2x_1} - F^1_{x_2}F^2_{x_1x_1} \\ F^1_{x_2}F^2_{x_3x_2} - F^1_{x_3}F^2_{x_2x_2} & \cdots & F^1_{x_1}F^2_{x_2x_2} - F^1_{x_2}F^2_{x_1x_2} \\ F^1_{x_2}F^2_{x_3x_3} - F^1_{x_3}F^2_{x_2x_3} & \cdots & F^1_{x_1}F^2_{x_2x_3} - F^1_{x_2}F^2_{x_1x_3} \end{bmatrix}$$
$$\ddot{\beta}(t) = \begin{bmatrix} T^*_{11} \\ T^*_{12} \\ T^*_{13} \end{bmatrix}^T \begin{bmatrix} F^1_{x_2x_1}F^2_{x_3} + F^1_{x_2}F^2_{x_3x_1} - F^1_{x_3x_1}F^2_{x_2} - F^1_{x_3}F^2_{x_2x_1} & \cdots & F^1_{x_1x_1}F^2_{x_2} + F^1_{x_1}F^2_{x_2x_1} - F^1_{x_2x_1}F^2_{x_1} - F^1_{x_2}F^2_{x_1x_1} \\ F^1_{x_2x_2}F^2_{x_3} + F^1_{x_2}F^2_{x_3x_2} - F^1_{x_3x_2}F^2_{x_2} - F^1_{x_3}F^2_{x_2x_2} & \cdots & F^1_{x_1x_2}F^2_{x_2} + F^1_{x_1}F^2_{x_2x_2} - F^1_{x_2x_2}F^2_{x_1} - F^1_{x_2}F^2_{x_1x_2} \\ F^1_{x_2x_3}F^2_{x_3} + F^1_{x_2}F^2_{x_3x_3} - F^1_{x_3x_3}F^2_{x_2} - F^1_{x_3}F^2_{x_2x_3} & \cdots & F^1_{x_1x_3}F^2_{x_2} + F^1_{x_1}F^2_{x_2x_3} - F^1_{x_2x_3}F^2_{x_1} - F^1_{x_2}F^2_{x_1x_3} \end{bmatrix}$$
$$\ddot{\beta}(t) = \begin{bmatrix} T^*_{11} \\ T^*_{12} \\ T^*_{13} \end{bmatrix}^T \begin{bmatrix} \left(F^1_{x_2}F^2_{x_3} - F^1_{x_3}F^2_{x_2}\right)_{x_1} & -\left(F^1_{x_1}F^2_{x_3} - F^1_{x_3}F^2_{x_1}\right)_{x_1} & \left(F^1_{x_1}F^2_{x_2} - F^1_{x_2}F^2_{x_1}\right)_{x_1} \\ \left(F^1_{x_2}F^2_{x_3} - F^1_{x_3}F^2_{x_2}\right)_{x_2} & -\left(F^1_{x_1}F^2_{x_3} - F^1_{x_3}F^2_{x_1}\right)_{x_2} & \left(F^1_{x_1}F^2_{x_2} - F^1_{x_2}F^2_{x_1}\right)_{x_2} \\ \left(F^1_{x_2}F^2_{x_3} - F^1_{x_3}F^2_{x_2}\right)_{x_3} & -\left(F^1_{x_1}F^2_{x_3} - F^1_{x_3}F^2_{x_1}\right)_{x_3} & \left(F^1_{x_1}F^2_{x_2} - F^1_{x_2}F^2_{x_1}\right)_{x_3} \end{bmatrix}$$
$$\ddot{\beta}(t) = \begin{bmatrix} T^*_{11} & T^*_{12} & T^*_{13} \end{bmatrix} \begin{bmatrix} (T^*_{11})_{x_1} & (T^*_{12})_{x_1} & (T^*_{13})_{x_1} \\ (T^*_{11})_{x_2} & (T^*_{12})_{x_2} & (T^*_{13})_{x_2} \\ (T^*_{11})_{x_3} & (T^*_{12})_{x_3} & (T^*_{13})_{x_3} \end{bmatrix}$$
$$\ddot{\beta}(t) = (\nabla F_1 \times \nabla F_2) * \nabla(\nabla F_1 \times \nabla F_2)$$
$$\ddot{\beta}(t) = T^* * \nabla(T^*)$$

The $\frac{d}{dt}(T^*) = T^* * \nabla (T^*)$ is valid for $T^* = \nabla F_1 \times ... \times \nabla F_{n-1}$.
We have $\frac{d}{dt}[\nabla (T^*)] = \nabla (\nabla (T^*)) * (T^*)^T$ and $\frac{d}{dt}[\nabla (\nabla (T^*))] = T^* * \nabla (\nabla (\nabla (T^*)))$.
The Eq. (7) is a consequence of

$$\begin{gathered}
\dddot{\beta}(t) = \frac{d}{dt}\left[\frac{d}{dt}\left[\dot{\beta}(t)\right]\right] \\
\dddot{\beta}(t) = \frac{d}{dt}\left[(T^*) * \nabla (T^*)\right] \\
\dddot{\beta}(t) = \frac{d}{dt}\left[(T^*)\right] * \nabla (T^*) + (T^*) * \frac{d}{dt}\left[\nabla (T^*)\right] \\
\dddot{\beta}(t) = (T^*) * \nabla (T^*) * \nabla (T^*) + (T^*) * \nabla (\nabla (T^*)) * (T^*)^T
\end{gathered}$$

The Eq. (8) is a consequence of

$$\begin{gathered}
\ddddot{\beta}(t) = \frac{d}{dt}\left[\dddot{\beta}(t)\right] \\
\ddddot{\beta}(t) = \frac{d}{dt}\left[(T^*) * \nabla (T^*) * \nabla (T^*) + (T^*) * \nabla (\nabla (T^*)) * (T^*)^T\right] \\
\ddddot{\beta}(t) = \frac{d}{dt}\left[(T^*) * \nabla (T^*) * \nabla (T^*)\right] + \frac{d}{dt}\left[(T^*) * \nabla (\nabla (T^*)) * (T^*)^T\right] \\
\vdots \\
\ddddot{\beta}(t) = \dddot{\beta}(t) * \nabla (T^*) + 3\ddot{\beta}(t) * \nabla (\nabla (T^*)) * (T^*)^T + (T^*) * \left[T^* * \nabla (\nabla (\nabla (T^*)))\right] * (T^*)^T
\end{gathered}$$

3.1 Curvature Formulas

Outer Product

Theorem 1 *(Goldman) First Curvature of n-1 implicit hypersurfaces*

$$k_1 = \frac{\left\| T^* \wedge \left[T^* * \nabla (T^*)\right] \right\|}{\left\| T^* \right\|^3} \tag{9}$$

Proof. The outer product of $\dot{\beta}(t) \wedge \ddot{\beta}(t)$is

$$\begin{aligned}
\dot{\beta}(t) \wedge \ddot{\beta}(t) &= v^3 k_1(t) \left(\overline{\mathbf{b}}_1(t) \wedge \overline{\mathbf{b}}_2(t)\right) \\
\left\| \dot{\beta}(t) \wedge \ddot{\beta}(t) \right\| &= \left\| v^3 k_1(t) \left(\overline{\mathbf{b}}_1(t) \wedge \overline{\mathbf{b}}_2(t)\right) \right\| \\
&= v^3 k_1(t) \left\| \overline{\mathbf{b}}_1(t) \wedge \overline{\mathbf{b}}_2(t) \right\|,
\end{aligned}$$

then $k_1 = \dfrac{\left\| T^* \wedge \left[T^* * \nabla (T^*)\right] \right\|}{\left\| T^* \right\|^3}$.

Theorem 2 *Second Curvature of n-1 implicit hypersurfaces*

$$k_2 = \frac{\left\| T^* \wedge \left[T^* * \nabla (T^*)\right] \wedge \left[T^* * \nabla (T^*) * \nabla (T^*) + T^* * \nabla (\nabla (T^*)) * (T^*)^T\right] \right\|}{\left\| T^* \right\|^6 k_1^2} \tag{10}$$

Proof. The outer product of $\dot{\beta}(t) \wedge \ddot{\beta}(t) \wedge \dddot{\beta}(t)$ is

$$\begin{aligned}
\dot{\beta}(t) \wedge \ddot{\beta}(t) \wedge \dddot{\beta}(t) &= v^6 k_1^2(t) k_2(t) \left(\overline{\mathbf{b}}_1(t) \wedge \overline{\mathbf{b}}_2(t) \wedge \overline{\mathbf{b}}_3(t)\right) \\
\left\| \dot{\beta}(t) \wedge \ddot{\beta}(t) \wedge \dddot{\beta}(t) \right\| &= \left\| v^6 k_1^2(t) k_2(t) \left(\overline{\mathbf{b}}_1(t) \wedge \overline{\mathbf{b}}_2(t) \wedge \overline{\mathbf{b}}_3(t)\right) \right\| \\
&= v^6 k_1^2(t) k_2(t) \left\| \overline{\mathbf{b}}_1(t) \wedge \overline{\mathbf{b}}_2(t) \wedge \overline{\mathbf{b}}_3(t) \right\|,
\end{aligned}$$

then $k_2 = \dfrac{\left\| T^* \wedge \left[T^* * \nabla (T^*)\right] \wedge \left[T^* * \nabla (T^*) * \nabla (T^*) + T^* * \nabla (\nabla (T^*)) * (T^*)^T\right] \right\|}{\left\| T^* \right\|^6 k_1^2}$.

Theorem 3 *Third Curvature of n-1 implicit hypersurfaces*

$$k_3 = \frac{\left\| T^* \wedge [T^* * \nabla (T^*)] \wedge \left[T^* * \nabla (T^*) * \nabla (T^*) + T^* * \nabla (\nabla (T^*)) * (T^*)^T \right] \wedge \overset{....}{\beta}(t) \right\|}{\|T^*\|^{10} k_1^3(t) k_2^2(t)}. \quad (11)$$

Proof. The outer product of $\dot{\beta}(t) \wedge \ddot{\beta}(t) \wedge \dddot{\beta}(t) \wedge \overset{....}{\beta}(t)$is

$$\begin{aligned} \dot{\beta}(t) \wedge \ddot{\beta}(t) \wedge \dddot{\beta}(t) \wedge \overset{....}{\beta}(t) &= v^{10} k_1^3(t) k_2^2(t) k_3(t) \left(\overline{\mathbf{b}}_1(t) \wedge \overline{\mathbf{b}}_2(t) \wedge \overline{\mathbf{b}}_3(t) \wedge \overline{\mathbf{b}}_4(t) \right) \\ \left\| \dot{\beta}(t) \wedge \ddot{\beta}(t) \wedge \dddot{\beta}(t) \wedge \overset{....}{\beta}(t) \right\| &= \left\| v^{10} k_1^3(t) k_2^2(t) k_3(t) \left(\overline{\mathbf{b}}_1(t) \wedge \overline{\mathbf{b}}_2(t) \wedge \overline{\mathbf{b}}_3(t) \wedge \overline{\mathbf{b}}_4(t) \right) \right\| \\ &= v^{10} k_1^3(t) k_2^2(t) k_3(t) \left\| \overline{\mathbf{b}}_1(t) \wedge \overline{\mathbf{b}}_2(t) \wedge \overline{\mathbf{b}}_3(t) \wedge \overline{\mathbf{b}}_4(t) \right\| \end{aligned}$$

Then $k_3 = \dfrac{\left\| T^* \wedge [T^* * \nabla (T^*)] \wedge \left[T^* * \nabla (T^*) * \nabla (T^*) + T^* * \nabla (\nabla (T^*)) * (T^*)^T \right] \wedge \overset{....}{\beta}(t) \right\|}{\|T^*\|^{10} k_1^3(t) k_2^2(t)}$.

4 Examples (The intersection $F_1 \cap F_2 \cap F_3 \cap F_4$ is Helix in $\mathbb{R}^3$)

Example 1 *The implicit surface* F_1 ,F_2, F_3 *and* F_4 *are given by* ,
$F_1(x, y, z, w, u) = u,$
$F_2(x, y, z, w, u) = w,$
$F_3(x, y, z, w, u) = x^2 + y^2 - w - u - a^2$
$F_4(x, y, z, w, u) = y - x \tan\left(\dfrac{z}{b}\right).$

The point of the intersection curve is $p_0 = (a\frac{\sqrt{2}}{2}, a\frac{\sqrt{2}}{2}, b\frac{\pi}{4}, 0, 0) \in S^{F_1} \cap S^{F_2} \cap S^{F_3} \cap S^{F_4}$. We have $\nabla F_1 = (0,0,0,0,1)$, $\nabla F_2 = (0,0,0,1,0)$, $\nabla F_3 = \left(a\sqrt{2}, a\sqrt{2}, 0, -1, -1\right)$ and $\nabla F_4 = \left(-1, 1, -\dfrac{a\sqrt{2}}{b}, 0, 0\right)$.

Compute $\dot{\beta}(t)$, $\ddot{\beta}(t)$, $\dddot{\beta}(t)$
Compute $\dot{\beta}(t)$

$\dot{\beta} = \nabla F_1 \times \nabla F_2 \times \nabla F_3 \times \nabla F_4$ where $\times$ is cross product.

$$\dot{\beta} = \left(\frac{2a^2}{b}, -\frac{2a^2}{b}, -2\sqrt{2}a, 0, 0 \right),$$

Compute $\ddot{\beta}(t)$
$\ddot{\beta} = T^* * \nabla T^*$ where $T^* = (T11, T12, T13, T14, T15)$

$$\nabla (T^*) = \begin{pmatrix} T11_x & T12_x & T13_x & T14_x & T15_x \\ T11_y & T12_y & T13_y & T14_y & T15_y \\ T11_z & T12_z & T13_z & T14_z & T15_z \\ T11_w & T12_w & T13_w & T14_w & T15_w \\ T11_u & T12_u & T13_u & T14_u & T15_u \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{2}a}{b} & -\frac{4\sqrt{2}a}{b} & -2 & 0 & 0 \\ \frac{2\sqrt{2}a}{b} & 0 & -2 & 0 & 0 \\ \frac{4a^2}{b^2} & -\frac{4a^2}{b^2} & -\frac{2\sqrt{2}a}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\ddot{\beta} = T^* * \nabla T^* = \begin{pmatrix} \frac{2a^2}{b} & -\frac{2a^2}{b} & -2\sqrt{2}a & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2\sqrt{2}a}{b} & -\frac{4\sqrt{2}a}{b} & -2 & 0 & 0 \\ \frac{2\sqrt{2}a}{b} & 0 & -2 & 0 & 0 \\ \frac{4a^2}{b^2} & -\frac{4a^2}{b^2} & -\frac{2\sqrt{2}a}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\ddot{\beta} = \left(-\frac{8\sqrt{2}a^3}{b^2}, 0, \frac{8a^2}{b}, 0, 0 \right)$$

Compute $\dddot{\beta}(t)$

$$\dddot{\beta}(t) = T^* * \nabla(T^*) * \nabla(T^*) + T^* * \nabla(\nabla(T^*)) * (T^*)^T$$

$$\nabla(\nabla T^*) =$$

$$= \left(\begin{pmatrix} T11_{xx} & T11_{xy} & T11_{xz} & T11_{xw} & T11_{xu} \\ T11_{yx} & T11_{yy} & T11_{yz} & T11_{yw} & T11_{yu} \\ T11_{zx} & T11_{zy} & T11_{zz} & T11_{zw} & T11_{zu} \\ T11_{wx} & T11_{wy} & T11_{wz} & T11_{ww} & T11_{wu} \\ T11_{ux} & T11_{uy} & T11_{uz} & T11_{uw} & T11_{uu} \end{pmatrix}, \cdots, \begin{pmatrix} T15_{xx} & T15_{xy} & T15_{xz} & T15_{xw} & T15_{xu} \\ T15_{yx} & T15_{yy} & T15_{yz} & T15_{yw} & T15_{yu} \\ T15_{zx} & T15_{zy} & T15_{zz} & T15_{zw} & T15_{zu} \\ T15_{wx} & T15_{wy} & T15_{wz} & T15_{ww} & T15_{wu} \\ T15_{ux} & T15_{uy} & T15_{uz} & T15_{uw} & T15_{uu} \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} 0 & \frac{4}{b} & \frac{4\sqrt{2}a}{b^2} & 0 & 0 \\ \frac{4}{b} & 0 & \frac{4\sqrt{2}a}{b^2} & 0 & 0 \\ \frac{4\sqrt{2}a}{b^2} & \frac{4\sqrt{2}a}{b^2} & \frac{16a^2}{b^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{8}{b} & 0 & -\frac{8\sqrt{2}a}{b^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{8\sqrt{2}a}{b^2} & 0 & -\frac{16a^2}{b^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{4}{b} & 0 & 0 \\ 0 & -\frac{4}{b} & -\frac{4\sqrt{2}a}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, [0], [0] \right), \text{ where}$$

$$[0] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ then } \dddot{\beta}(t) = \left(\frac{96a^4}{b^3}, 0, -\frac{64\sqrt{2}a^3}{b^2}, 0, 0 \right).$$

Compute $\ddddot{\beta}(t)$

$$\ddddot{\beta}(t) = \dddot{\beta}(t) * \nabla(T^*) + 3\ddot{\beta}(t) * \nabla(\nabla(T^*)) * (T^*)^T + (T^*) * [T^* * \nabla(\nabla(\nabla(T^*)))] * (T^*)^T$$

$$\nabla(\nabla(\nabla(T^*))) = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} \end{bmatrix}, J_{1k} = \nabla^T(H_{1k}) = \begin{bmatrix} (H_{1k})_{x_1} \\ (H_{1k})_{x_2} \\ (H_{1k})_{x_3} \\ (H_{1k})_{x_4} \\ (H_{1k})_{x_5} \end{bmatrix},$$

$$H_{1k} = \nabla J_{1k} = \begin{bmatrix} \nabla\left((T^*_{1k})_{x_1}\right) \\ \nabla\left((T^*_{1k})_{x_2}\right) \\ \nabla\left((T^*_{1k})_{x_3}\right) \\ \nabla\left((T^*_{1k})_{x_4}\right) \\ \nabla\left((T^*_{1k})_{x_5}\right) \end{bmatrix} = \begin{bmatrix} (T^*_{1k})_{x_1x_1} & (T^*_{1k})_{x_1x_2} & (T^*_{1k})_{x_2x_3} & (T^*_{1k})_{x_1x_4} & (T^*_{1k})_{x_1x_5} \\ (T^*_{1k})_{x_2x_1} & (T^*_{1k})_{x_2x_2} & (T^*_{1k})_{x_2x_3} & (T^*_{1k})_{x_2x_4} & (T^*_{1k})_{x_2x_5} \\ (T^*_{1k})_{x_3x_1} & (T^*_{1k})_{x_3x_2} & (T^*_{1k})_{x_3x_3} & (T^*_{1k})_{x_3x_4} & (T^*_{1k})_{x_3x_5} \\ (T^*_{1k})_{x_4x_1} & (T^*_{1k})_{x_4x_2} & (T^*_{1k})_{x_4x_3} & (T^*_{1k})_{x_4x_4} & (T^*_{1k})_{x_4x_5} \\ (T^*_{1k})_{x_5x_1} & (T^*_{1k})_{x_5x_2} & (T^*_{1k})_{x_5x_3} & (T^*_{1k})_{x_5x_4} & (T^*_{1k})_{x_5x_5} \end{bmatrix},$$

$$\nabla(\nabla(\nabla(T^*))) = \begin{bmatrix} (H_{11})_{x_1} & (H_{12})_{x_1} & (H_{13})_{x_1} & (H_{14})_{x_1} & (H_{15})_{x_1} \\ (H_{11})_{x_2} & (H_{12})_{x_2} & (H_{13})_{x_2} & (H_{14})_{x_2} & (H_{15})_{x_2} \\ (H_{11})_{x_3} & (H_{12})_{x_3} & (H_{13})_{x_3} & (H_{14})_{x_3} & (H_{15})_{x_3} \\ (H_{11})_{x_4} & (H_{12})_{x_4} & (H_{13})_{x_4} & (H_{14})_{x_4} & (H_{15})_{x_4} \\ (H_{11})_{x_5} & (H_{12})_{x_5} & (H_{13})_{x_5} & (H_{14})_{x_5} & (H_{15})_{x_5} \end{bmatrix},$$

$$\nabla(\nabla(\nabla(T^*))) =$$

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{b^2} & 0 & 0 \\ 0 & \frac{8}{b^2} & \frac{16\sqrt{2}a}{b^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & -\frac{16}{b^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{16}{b^2} & 0 & -\frac{32\sqrt{2}a}{b^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & [0] & [0] \\ \begin{bmatrix} 0 & 0 & \frac{8}{b^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{8}{b^2} & 0 & -\frac{32\sqrt{2}a}{b^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{8}{b^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & [0] & [0] \\ \begin{bmatrix} 0 & \frac{8}{b^2} & \frac{16\sqrt{2}a}{b^3} & 0 & 0 \\ \frac{8}{b^2} & 0 & \frac{16\sqrt{2}a}{b^3} & 0 & 0 \\ \frac{16\sqrt{2}a}{b^3} & \frac{16\sqrt{2}a}{b^3} & \frac{80a^2}{b^4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -\frac{16}{b^2} & 0 & -\frac{32\sqrt{2}a}{b^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{32\sqrt{2}a}{b^3} & 0 & -\frac{80a^2}{b^4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{8}{b^2} & 0 & 0 \\ 0 & -\frac{8}{b^2} & -\frac{16\sqrt{2}a}{b^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] \end{bmatrix}$$

where $[0] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\ddddot{\beta}(t) = \left(\frac{384\sqrt{2}a^5}{b^4}, \frac{128\sqrt{2}a^3}{b^2}, \frac{128(12a^6 + 11a^4b^2)}{b^5}, 0, 0 \right).$$

Compute Curvatures $k_1(t)$, $k_2(t)$ and $k_3(t)$

Compute for using outer product

Compute $k_1(t)$

$$\|u_1 \wedge u_2\| = \sqrt{\sum_J \det(\alpha^J)^2} = \sqrt{\sum_{j_1<j_2} \begin{vmatrix} \alpha_1^{j_1} & \alpha_2^{j_1} \\ \alpha_1^{j_2} & \alpha_2^{j_2} \end{vmatrix}^2} = \sqrt{\|u\|^2 \|v\|^2 - (u \cdot v)^2},$$

where $\{1 \leq j_1 \leq \cdots \leq j_2 \leq \cdots \leq 5\}$

$$k_1 = \frac{\|T^* \wedge [T^* * \nabla(T^*)]\|}{\|T^*\|^3} = \tfrac{a}{a^2+b^2}$$

Compute $k_2(t)$

$$\|u_1 \wedge u_2 \wedge u_3\| = \sqrt{\sum_J \det(\alpha^J)^2} = \sqrt{\sum_{j_1<j_2<j_3} \begin{vmatrix} \alpha_1^{j_1} & \alpha_2^{j_1} & \alpha_3^{j_1} \\ \alpha_1^{j_2} & \alpha_2^{j_2} & \alpha_3^{j_2} \\ \alpha_1^{j_3} & \alpha_2^{j_3} & \alpha_3^{j_3} \end{vmatrix}^2},$$

where $\{1 \leq j_1 \leq \cdots \leq j_2 \leq \cdots \leq j_3 \leq 5\}$

$$k_2 = \frac{\left\|\dot{\beta}(t) \wedge \ddot{\beta}(t) \wedge \dddot{\beta}(t)\right\|}{v^6 k_1^2} = \tfrac{b}{a^2+b^2}$$

Compute $k_3(t)$

There $\wedge$ is outer product of the four vectors in (5) $-$dimensional is the same $\times$ is cross product of the four vectors in (5) $-$dimensional.

$$k_3 = \frac{\left\|\dot{\beta}(t) \wedge \ddot{\beta}(t) \wedge \dddot{\beta}(t) \wedge \ddddot{\beta}(t)\right\|}{\|T^*\|^{10} k_1^3(t)\, k_2^2(t)} = \frac{\left\|\dot{\beta}(t) \times \ddot{\beta}(t) \times \dddot{\beta}(t) \times \ddddot{\beta}(t)\right\|}{\|T^*\|^{10} k_1^3(t)\, k_2^2(t)} = 0$$

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