

A Graph-based adaptive triangular-mesh refinement applied to classical elliptical and parabolical problems

Sanderson L. Gonzaga de Oliveira,

CEAVI, UDESC,
89140-000, Ibirama, SC
E-mail: sgonzaga@ic.uff.br,

Mauricio Kischinhevsky

IC - UFF
24210-240, Niterói, RJ
E-mail: kisch@ic.uff.br.

Abstract: *This work proposes an adaptive triangular-mesh refinement based on the Finite Volume Method with cell-centered control volumes in order to numerically solve second order partial differential equations. The mesh is represented by a graph data structure in which parent nodes of the refined ones, in the adaptive mesh refinement process, are not stored. The scheme intends to reduce the computational cost to construct the refined mesh in evolutionary problems. Additionally, linear-system solvers based on the minimization of functionals can be easily employed. Specifically, the Conjugate Gradient Method is used. Numerical solutions of an elliptical and a parabolical problems demonstrate the efficiency and advantages of this scheme.*

Keywords: *Adaptive mesh refinement, Finite Volume Method, Partial Differential Equations, Conjugate Gradient Method.*

1 Introduction

Numerical solution of partial differential equations may require the use of a mesh refinement strategy that concentrates more mesh points where the solution and/or its derivatives change rapidly. For time-dependent problems, adaptive methods become particularly important since, by the dynamic nature of such problems, may happen migration (or occurrence) of regions that have rapid solution change.

This work represents a triangular-discretized domain by a graph data structure. Triangular meshes, or triangulations, are one of the most widely used representations for geometric models. A triangulation is a 2D simplicial complex, a simple structure with convenient combinatorial properties such as cited by [5]. A non-uniform mesh comprised of triangular control volumes may be more appropriated near physical boundary regions and features of a problem with complex geometries.

This present technique operates all the required elements for the adaptive triangular-mesh refinement for the Finite Volume Method. Triangular meshes are shown in experimental tests in order to exemplify this technique. Moreover, this work presents tests of the discrete adaptive mesh comprised by triangular control volumes following the Autonomous Leaves Graph concepts [1].

After this brief introduction, Section 2 treats the graph data structure. Afterwards, Section 3 discusses some experimental tests. Finally, Section 4 draws some considerations.

2 Graph-based adaptive mesh refinement technique with triangular control volumes

This work uses a graph-based adaptive triangular-mesh refinement that enables all the requirements for the solution of partial differential equations based on the Finite Volume Method with triangular control volumes.

The graph enables all the relations among control volumes. The nodes correspond to the control volumes. Two nodes are connected if their associated volumes have a common edge. Furthermore, when a control volume is refined, a parent control volume node is substituted by a new sub-graph, i.e. a pack with four control volume nodes and three transition nodes. Transition nodes indicate the refinement level of the control volume in relation to their neighbor control volumes. The refinement process is sketched in Fig. 1, whose most right sub-graph is a pack that regenerates its parent control volume node on its left-hand side. Black circles represent the control nodes and white circles represent the transition nodes of the graph.

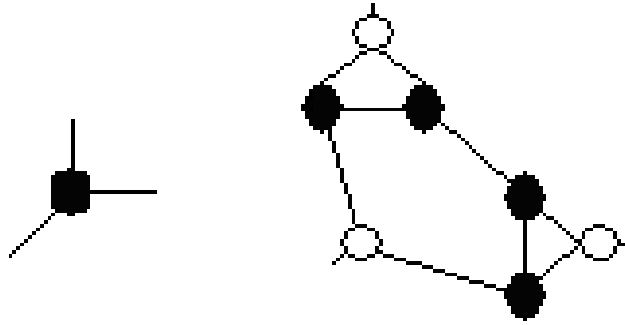


Figure 1: Right-hand side sub-graph created after a refinement of the left-hand side parent control volume node

Only four control volume graph nodes that represent the four new control volumes and the three required transition graph nodes are stored in the new pack. Parent nodes are deleted in the local refinement of each triangular control volume.

Left-hand side of Fig. 2 depicts an example of an initial discretization represented by a unit square and the boundaries of the domain are neglected for clarity. Besides, the barycenters of the triangles are represented by black points. Right-hand side of Fig. 2 depicts a graph that represents this initial discretization. Control volume graph nodes in right-hand side of Fig. 2 represent each control volume of the left-hand side of Fig. 2. Graph links are represented by lines.

Figure 3 represents a refined control volume from the left-hand side of Fig. 2. In this figure, the domain boundaries are also neglected for clarity. The highlighted new pack in the graph in the right-hand side of Fig. 3 is generated by the mesh refinement process showed in its right-hand side.

This solution scheme of partial differential equations produces linear systems. The discrete places must be numbered so that linear systems have each row corresponding to a specific one. The ordering of all nodes is a step which can be carried out in several ways. In this work, numbering is performed by a modified Sierpiński-like covering. Thus, this work applies a modified Sierpiński-like space-filling Curve for total ordering of the graph-based triangular-mesh representation [2]. Moreover, the Conjugate Gradient Method [4] is employed, which is one of the most prominent iterative method for solving sparse systems of linear equations.

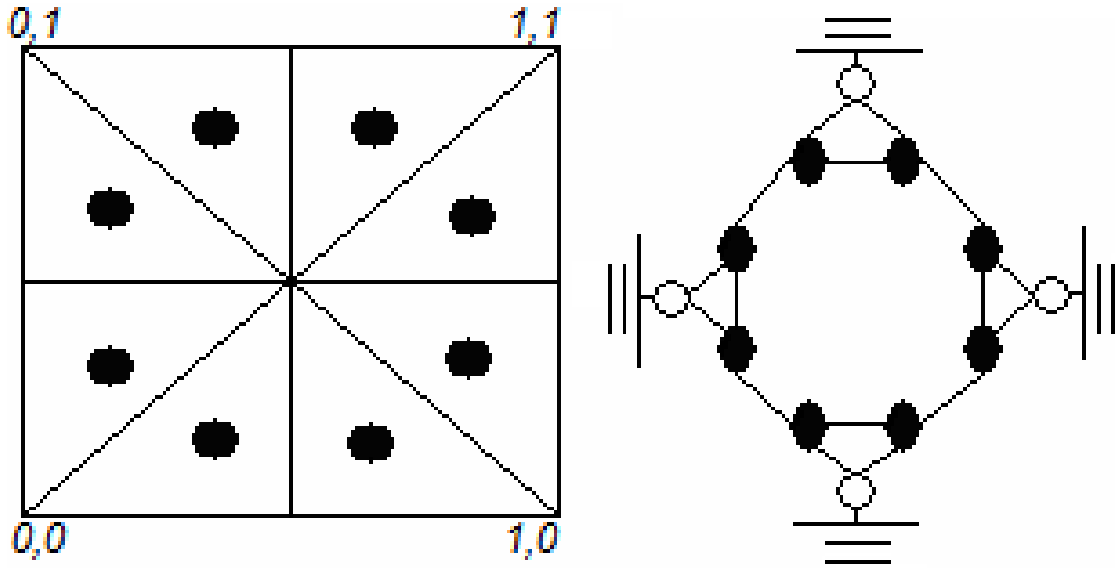


Figure 2: Discretized unit square as the problem domain and its graph data structure representation, respectively, where nodes represent the refinement level 0

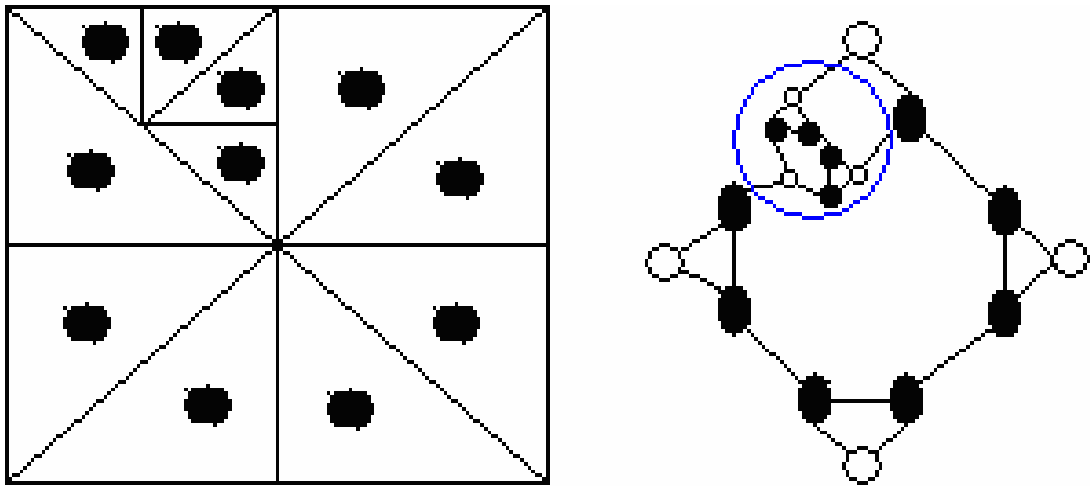


Figure 3: Example of a refined control volume and a highlighted new pack in the graph generated by the refinement process, respectively

3 Experimental tests

This scheme is applied to two classical problems, i.e. the Laplace Problem and the Heat Conduction Equation. The first one is a stationary elliptical problem. The second is an evolutionary parabolical one. These tests exemplify the use of this technique.

3.1 Laplace Equation

Consider the Dirichlet problem given by

$$\begin{aligned} \nabla^2 \phi &= 0 \text{ in } \Omega, \\ \phi &= f \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

where ϕ is the dependent variable of the partial differential equation, Ω is a limited domain in \mathbb{R}^2 and f is a defined smooth function on a boundary $\partial\Omega$. In 1902, Jacques Salomon Hadamard

(1865-1963) claimed that, for a mathematical problem correspond to reality, the following basic conditions should be satisfied: the solution must exist (existence); the solution must be determined by data of a unique form (unicity); and the solution must depend on data of continual form (stability). Thus, problem (1) is well-posed in the sense of Hadamard [6].

Consider the numerical solution of (1) in a unit square in an implicit formulation through the Finite Volume Method basic formulation for irregular meshes

$$\int_{v_i} \nabla \phi \cdot \vec{n} d(\partial v_i) = 0 \quad (2)$$

where \vec{n} is the normal outward vector of the control volume v_i and ∂v_i represents the boundary of the control volume v_i . Furthermore, top, bottom and left sides of the unit square present prescribed boundary condition with a unique constant f . On the other hand, right side of the unit square presents a different value for f . The scheme adopted by [3] is applied in order to solve the gradient of (2) in the edge of the control volumes. Figure 4 illustrates the final mesh configuration using a maximum of eight levels of refinement for each cell. The mesh is comprised of 4421 volumes, whose Conjugate Gradient Method converges after 729 iterations.

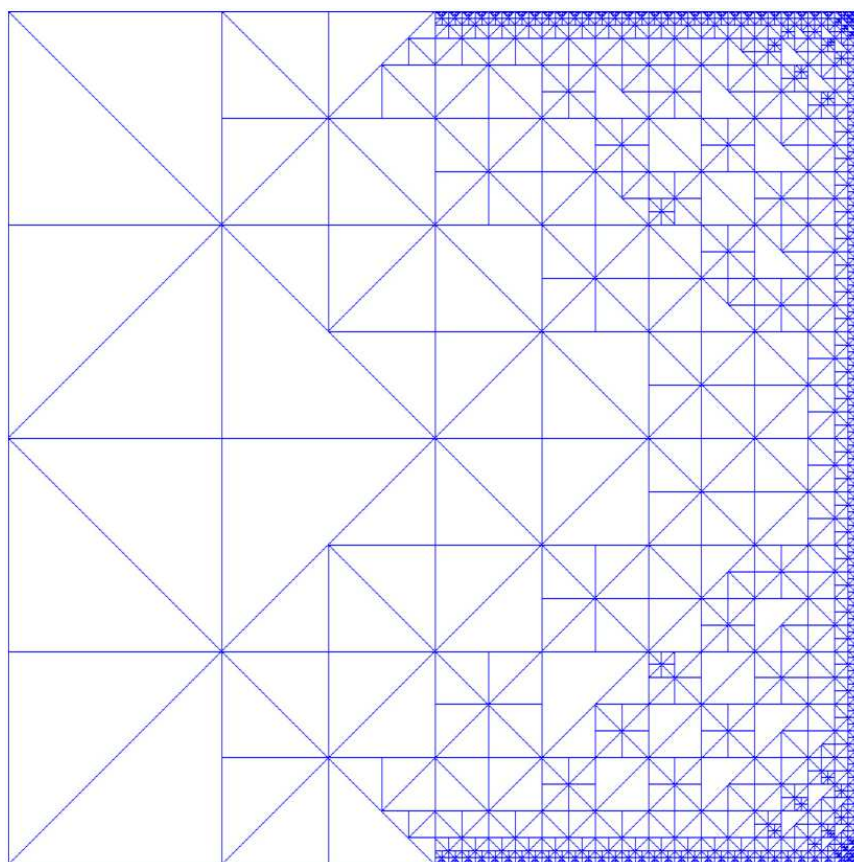


Figure 4: An adaptive refinement mesh of the Laplace equation

3.2 Heat Conduction Equation

Consider the numerical solution of the 2D heat conduction equation problem

$$\begin{aligned} \phi_t &= \nabla^2 \phi \\ \phi(u, 0) &= f(u), \quad u \in \Omega, \quad f(u) \equiv 0, \\ \phi(u, t) &= g(u), \quad u \in \partial\Omega, \quad t \geq 0, \end{aligned} \tag{3}$$

where $\Omega \subset \mathbb{R}$, thus, $u = (x, y)$. Function f is a smooth function limited in Ω . Problem (3) is also well-posed in the sense of Hadamard [6].

Consider the numerical solution of (3) in a unit square through an implicit formulation following the Finite Volume Method basic formulation for irregular meshes

$$\frac{M\phi_p^{n+1}}{\Delta t} - \nabla^2 \phi_p^{n+1} = \frac{M\phi_p^n}{\Delta t} \tag{4}$$

where M represents the area of triangle p . These tests applied $\Delta t = 0.1$. Boundary conditions on top, bottom and left sides of the unit square have a unique prescribed boundary value f and the right side has a different value.

Figure 5 illustrates the final mesh configuration after ten time steps and seven maximum refinement levels for each volume. The mesh is comprised of 1496 volumes, whose Conjugate Gradient Method converges in 519 iterations.

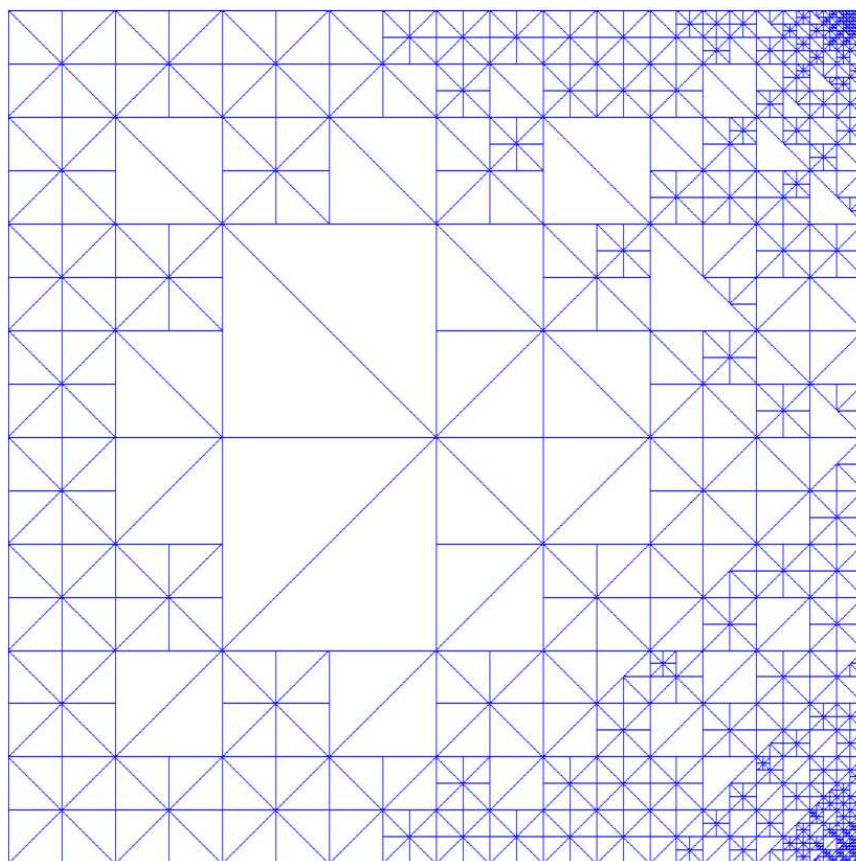


Figure 5: Final mesh configuration of an adaptive triangular-mesh refinement approximation to the Heat Conduction Equation.

4 Conclusions

This work proposes a triangular discretization based on the Finite Volume Method for solving second order partial differential equations with irregular meshes. The required data for solving a partial differential equation are stored in a graph data structure that performs the adaptive triangular-mesh refinement.

This work shall continue in order to develop a computational implementation of more complex problems. In addition, since tests were applied in 2D, new experiments shall apply the proposed scheme in 3D. This scheme must be compared to other techniques in order to test its performance and accuracy. Those tests may demonstrate the efficiency of this scheme and its advantages in comparison to existing techniques applied in adaptive triangular-mesh refinement techniques.

Acknowledgements

The authors gratefully acknowledge CAPES for the financial support.

References

- [1] D.D. Burgarelli, M. Kischinhevsky and R.J. Biezuner, A new adaptive mesh refinement strategy for numerically solving evolutionary PDE's. *Journal of Computational and Applied Mathematics*, 196 (2006) 115-131.
- [2] S.L. Gonzaga de O. and M. Kischinhevsky, Sierpinski curve for total ordering of a graph-based adaptive simplicial-mesh refinement for Finite Volume discretizations, *XXXI CNMAC* (2008).
- [3] S.L. Gonzaga de O., "Graph-based adaptive simplicial-mesh refinement for finite volume discretizations", Thesis, IC-UFF, 2009.
- [4] M.R. Hestenes and E. Stiefel, Methods of conjugate gradients for solving linear systems. *J.Res.Nat.Bur.Stand.*, 49 (1952) 409-436.
- [5] L. Velho, L.H. Figueiredo and J. Gomes, Hierarchical generalized triangle strips, *The Visual Computer*, 15 (1999) 1 21-35.
- [6] E. Zachmanoglou and D.W. Thoe, "Introduction to partial differential equations with applications", Courier Dover Publications, 1986.