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Linguistic Variables of Type-N.
A Mathematical Model

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Abstract: *This work proposes a model for linguistic variables and the fuzzyfication process for fuzzy systems which deals with different level of uncertainty in the same linguistic variables. Those systems are called here* ***Systems of Type-n****. We introduce concepts like n-***Homogeneous linguistic variables*** *which emphasises the ocurrence of different levels of uncertainty in such Fuzzy systems. The most important result is the statement of the* ***General Fuzzification Expression (GFE)*** *which establishes how the fuzzification process must be calculated in every system of type-n.*

Keywords: *Fuzzy Sets, Type-n Fuzzy sets, Systems of Type-n, n-homogeneous linguistic variables, heterogeneous linguistic variables.*

1 Introduction

Fuzzy expert systems and Fuzzy Control Systems usually deal with uncertainty as values in the set $[0, 1]$*. However, the concept of Type-n fuzzy sets leads to the possibility of different levels of uncertainty for elements of the same universe. A fuzzy system which deals with such level difference must be able to apply the calculation of the fuzzification process on different types of objects, in other words there must exist operations of coersion on the levels of uncertainties to enable the application of a fuzzification expression. Aiming to give a rigorous semantics for such situation, in this paper we propose a mathematical model for linguistic variables with possibly different levels of uncertainty (heterogeneous linguistic variables) and define a fuzzification expression which can be applied both to homogeneous and heterogeneous linguistic variables.*

2 Linguistic Variables

First we give one classical definition for linguistic variables which can be found in [9]:

Definition 2.1 *A linguistic variable is a quintuple* $\langle x, T(x), U, G, M\rangle$*; where:* x *is the name of the variable;* $T(x)$ *is a set of terms which are values for the variable;* U *is the universe of discourse which defines the characteristics of the variable;* G *is a syntatic grammar which produces the terms in* $T(x)$*;* M *semantic rules which maps terms in* $T(X)$ *to fuzzy sets in* U*.*

Linguistic variables are objects which allow the association between names and elements of an universe, together with a confidence about that. In many applications, especially in Control Systems, linguistic variables are used to associate names to some notion of quantity. For example, a linguistic variable with names like "Small", "Medium", etc (which denote quantities) associated with real numbers which denote a measure of, for example, height. But observe in the previous definition that the names do not restrict themselves to make reference to quantities. In fact, linguistic variables can also be used in expert systems where terms denote objects with no relation to quantities. In this work we propose a definition for linguistic variables which is a little extension from the previous one, we do that to specify more accurately the concept of fuzzification and situations where the notion of fuzzyness is not an element in $[0, 1]$.

Since linguistic variables state the association between terms and elements of the universe with a degree of confidence, the concept of confidence makes part of the definition of linguistic variables. Therefore, it is worthy to note that the notion of confidence or uncertainty as values in the interval $[0, 1]$ *can be extended. In the literature the corresponding concept for that is the notion of* ***type-n fuzzy set***. *The idea is: If the grade of membership* $f(x)$ *is a value in* $[0, 1]$, *the corresponding fuzzy set "f" is called of* ***type-1***, *if the grade is a type-1 fuzzy set, then the corresponding fuzzy set is of* ***type-2***, *and so on. Here we make the convention that an element* $r \in [0, 1]$ *is also called a fuzzy set of* ***type-0***. *Hence, the concept of type-n fuzzy sets introduces levels of uncertainty or confidence, which must be taken into account in the definiton of linguistic variable.*

Definition 2.2 (First Alternate Definition) *A* linguistic variable $\mathcal{L}$ *is a structure*

$$\mathcal{L} = \langle N, U, G, T_G, M_G \rangle \quad (1)$$

where:

1. N *is a string designating the name of the linguistic variable,*
2. U *is the universe of discourse which defines the characteristics of the variable;*
3. G *is a grammar;*
4. T_G *is a fuzzy set, whose elements are the strings generated by* G *together with a degree of confidence; i.e.* T_G *is a function of the form* $T_G : L(G) \rightarrow \mathbb{FZSET}$, *where* $L(G)$ *is the formal language generated by* G *and* $\mathbb{FZSET}$ *is the class of all type-0, type-1, etc fuzzy sets. The elements of* $L(G)$ *are called* ***linguistic values*** *or* ***linguistic terms***, *while the elements of the fuzzy set* T_G *are called* ***fuzzy terms***.
5. M_G *is a, possibly empty, function* $M_G : L(G) \rightarrow U \rightarrow \mathbb{FZSET}$. *It gives a degree of possibility for the relation between each linguistic term and each element* $u \in U$.

The ***class of all linguistic variables*** *is denoted here by* $\mathbb{LVAR}$. *The constructors of* G *with arity greater than zero are called* ***hedges***.

3 Fuzzification in Systems of Type-0

To fuzzify has two meanings: (1) to transform a classical concept into a fuzzy concept; and (2) according to an input A, *to find the confidence of fuzzy terms; i.e. the values* $T_G(x)$, *for every* $x \in L(G)$. *We apply fuzzification in the second sense.*

According to [13] p.51:

> *"At the end of a sequence of rule firings in a fuzzy expert system we may end up with a* FUZZY CONCLUSION C THAT IS A LINGUISTIC VARIABLE, WHOSE VALUES HAVE BEEN ASSIGNED GRADES OF MEMBERSHIP."

In what follows we propose a model for the process of fuzzification which is the process that assigns grades of membership to the values of linguistic variables (to the linguistic terms). The proposed model is a function that transforms linguistic variables plus a fuzzy set into a linguistic variables. At this point we model fuzzification for the systems that make associations between each $(t,u) \in L(G) \times U$ *and values at* $[0,1]$*, which are called here:* ***systems of type-0****. This notion will be later generalized, in order to model systems which assign other (possible different) levels of uncertainty to each* $(t,u) \in L(G) \times U$*.*

Definition 3.1 *A* ***system of Type-0*** *is a system such that for every linguistic variable* $\mathcal{L}, t \in L(G)$*, and* $u \in U$*, we have* $M_G(t)(u) \in [0,1]$*.*

Proposition 3.2 *Given a set* B *and an element* $b \in B$*, there is a fuzzy set* $Sing_b^B : B \to [0,1]$*, such that* $Sing_b^B(b) = 1$ *and* $Sing_b^B(x) = 0$*; for every* $x \neq b$*.*

Demonstration *Trivial.* ***Q.E.D.***

Definition 3.3 *The fuzzy set* $Sing_b^B$ *is called the* ***singleton associated with*** b ***in*** B*.*

Definition 3.4 (Type-0 Fuzzified Linguistic Variables) *Given a system of Type-0,* S*, a linguistic variable* $\mathcal{L} = \langle N, U, G, T_G, M_G \rangle$ *of* S *and a fuzzy set* $A : U \to [0,1]$*, a linguistic variable* $\mathcal{L}_A = \langle N, U, G, T_{G_A}, M_G \rangle$ *is a* ***fuzzified linguistic variable from*** A ***and*** $\mathcal{L}$*, if for all* $t \in L(G)$ *and* $x \in U$*,*

$$T_{G_A}(t) = max(min(A(x), M_G(t)(x))) \tag{2}$$

Corollary 3.5 *This definition lead us to the following abbreviation:*

$$T_{G_A}(t) = \begin{cases} M_G(t)(a) & \text{, if } A = Sing_a^U\text{; or} \\ max(min(A(x), M_G(t)(x))) & \text{for all } x \in U\text{, otherwise.} \end{cases} \tag{3}$$

Corollary 3.6 *In a system of type-0, for every linguistic variable* $\mathcal{L}$*,* T_G *in* $\mathcal{L}$*, and for every* $t \in T_G$*, the fuzzy sets* A *and* T_{G_A} *are functions of the form* $A : U \to [0,1]$ *and* $T_{G_A} : L(G) \to [0,1]$*.*

Demonstration *Straightforward from equation (2).* ***Q.E.D.***

Considering the infimum operation "$\wedge$*" as the pointwise extension of the minimum over* $[0,1]$ *to the class of type-1 fuzzy sets* $[U \to [0,1]]$*, the equation 2 can be rewritten as:*

$$T_{G_A}(t) = max\{(M_G(t) \wedge A)(u) | u \in U\} \tag{4}$$

According to corollary 3.6, systems of type-0 always return linguistic variables which the updated values of confidence is in $[0,1]$*. Moreover, if we look more closely at the functions that make part of the fuzzification process; namely:* M_G*,* T_G*,* A*, and* $(M_G(t) \wedge A)$*, they have the following types:* $M_G : L(G) \to U \to [0,1]$*,* $A : U \to [0,1]$*,* $T_G : L(G) \to [0,1]$*, and for each* $t \in L(G)$*,* $(M_G(t) \wedge A) : U \to [0,1]$*. In other words, for systems of type-0, the fuzzification expressions involve the functions* $M_G : L(G) \to U \to [0,1]$ *and* $A : U \to [0,1]$*, and yields a function* $T_{G_A} : L(G) \to [0,1]$*. So the fuzzy sets, in the type* $\mathbb{FZSET}$*, which are values of* $T_G(t)$ *and* $M_G(t)(u)$*, according to definition 2.2, are type-0 fuzzy sets; namely just values in* $[0,1]$*.*

In order to view a value as a function or to lift a type-0 fuzzy set to a type-1 fuzzy set as we will do later on it is possible to do constant lifting as follows.

Proposition 3.7 *Given any set* B *and an element* $r \in [0,1]$*, there is a fuzzy set* $C_r : B \to [0,1]$*, such that* $C_r(b) = r$*; for every* $b \in B$*.*

The set B *can be a singleton set or a big product type as we will see later.*

The questions now are:

1. *What is the type of the functions T_G, M_G and A, for general systems which allow different levels of uncertainty for each $u \in U$? and*
2. *how the expression of fuzzification looks like?*

The following section deals with such questions, and based on the known notion of fuzzy sets of type-n, it introduces the notion of System of type-n and generalizes equation 2 to deal with different levels of uncertainty at the same linguistic variable. It is done in a such a way that systems of type-0 is a special case.

4 Systems of Type-n

Definition 4.1 (Type-n Linguistic Variables and Systems of Type-n) *A **Type-n Linguistic Variable**, $\mathcal{L}$, abbreviated by $\tau(\mathcal{L}) = n$, is a linguistic variable such that there are $u \in U$ and $t \in L(G)$, where $M_G(u)(t)$ is a type-n fuzzy set, and such that for any other type-k fuzzy set $M_G(u')(t')$, where $u' \in U$ and $t' \in L(G)$, $k \leq n$. A Fuzzy system is called a **Fuzzy System of Type-n**, if for all of its linguistic variables $\mathcal{L}_1, \ldots, \mathcal{L}_k$, $max(\tau(\mathcal{L}_1), \ldots, \tau(\mathcal{L}_k)) = n$. A linguistic variable is n**-homogeneous** if $M_G(u)(t)$ is a type-n fuzzy set, for every $u \in U$; otherwise it is called **heterogeneous**. A system is called n-homogeneous if every linguistic variable of it is n-homogeneous; otherwise it is called **heterogeneous**.*

In what follows, we show that for every type-n fuzzy set there is an equivalent type-1 fuzzy set, consequently, every system of type-n can be reduced to a system of type-1.

Lemma 4.2 *Given a type-n fuzzy set $f : A_1 \to (A_2 \to (\ldots(A_n \to [0,1])\ldots))$, also written without parentheses as $f : A_1 \to A_2 \to \ldots A_n \to [0,1]$, there is an equivalent fuzzy set:*

$$UC(f) : A_1 \times A_2 \times \ldots A_n \to [0,1] \tag{5}$$

Demonstration *Since $\mathbb{SET}$ is a cartesian closed category, $UC(f)(a_1, \ldots, a_n)$ is the uncurried version of f.* ***Q.E.D.***

Corollary 4.3 *Every type-n fuzzy set ($n > 1$) can be replaced by its uncurried counterpart "$UC(f)$"; which is a type-1 fuzzy set.*

Proposition 4.4 *For every type-n fuzzy set, $n \in \mathbb{N}$, there is an equivalent type-1 fuzzy set.*

Demonstration *It is enough to proof for $n = 0$, for we have previously proved for $n > 1$. According to proposition 3.7, every real number $r \in [0,1]$ can be functionally represented by a function C_r.* ***Q.E.D.***

Since the notion of uncertainty/confidence is not restricted to the set $[0,1]$, it is fairly possible to have a system with some linguistic variable such that for some $t \in L(G)$ and $u \in U$, $M_G(t)(u)$ are not fuzzy sets of the same level, i.e. functions of the same type; for example, it is possible that for some $(s,v) \in L(G) \times U$ the associated notion of uncertainty, $M_G(s)(v)$, be a type-0 fuzzy set, i.e. a number at $[0,1]$, and some other $(t,u) \in L(G) \times U$, $M_G(t)(u)$ is a type-1 fuzzy set. The questions here are:

1. *"How the functions M_G and A look like?"*
2. *"How do we calculate the fuzzification in such situation?"*
3. *"For each linguistic variable $\mathcal{L}$, what is the canonical type of T_G?"*

Analysing the expression "$T_{G_A}(t) = max(min(A(x), M_G(t)(x)))$" at the equation of fuzzification, eq (2) , we can observe some facts:

1. *the expression "$min(A(x), M_G(t)(x))$" imposes that $A(x)$ and $M_G(t)(x)$ should have the same type.*

2. *The type of $M_G(t)(x)$ only depends on $x \in U$, because for any two $t_1, t_2 \in L_G$ the types of $M_G(t_1)(x)$ and $M_G(t_2)(x)$ must be identical to $A(x)$.*

3. *The previous observation imposes a generalization of min operation to infimum operations (written as $\wedge$). Therefore, an equation like (4) is required.*

4. *The external "max" function imposes coersion operations on its operands.*

5. *The previous observations imposes a generalization of max operation to supremum operations (written as $\vee$).*

Therefore, equation (2) must be generalized to the following equation (6).

$$T_{G_A}(t) = \bigvee_{u \in U} (M_G(t)(u) \wedge A(u)) \tag{6}$$

Likewise, the equivalent equation (4) generalizes to the following equivalent equation:

$$T_{G_A}(t) = \bigvee_{u \in U} [(M_G(t) \wedge A)(u)] \tag{7}$$

Previously, the item 4 does not reveal another possible kind of required coersion to evaluate fuzzification. In order to simplify the notation, the infimum expression "$(A \wedge M_G(t))(x)$" will be denoted by M_x^t.

Since every fuzzy set can be expressed as type-1 fuzzy set, see proposition 4.4, it is enough to state the following results in terms of type-1 fuzzy sets. We assume that every type-0 fuzzy set $r \in [0,1]$ is transformed into a type-1 fuzzy set $C_r : \{*\} \rightarrow [0,1]$ — where $\{*\}$ is a singleton and a special case of proposition 3.7 — and every type-n fuzzy set, where $n > 1$, is in its uncurrified form.

It is not difficult to note that every function $M_G(t)(u)$ has a connection with its domain. But, if you look closely to the fuzzification process, the infimum expression states that the same function $A(u)$ is used as operand together with every function $M_G(t)(u)$. It means that the referred connection depends just on U. So, each function $M_G(t)(u)$ is associated with a type (its domain) P_u, no matter what $t \in L(G)$ is. Therefore, for each $u \in U$, there is a family of functions:

$$\gamma(u) = \{M_G(t)(u), A(u), M_u^t : P_u \rightarrow [0,1]\}. \tag{8}$$

Proposition 4.5 *Given a linguistic variable $\mathcal{L} = \langle N, U, G, T_G, M_G \rangle$, there is a family of functions:*

$$\overline{\gamma}(u) = \{\overline{M_G}(t)(u), \overline{A}(u), \overline{M}_u^t : P \rightarrow [0,1]\}. \tag{9}$$

such that $\overline{M_G}(t)(u), \overline{A}(u)$, and $\overline{M}_u^t$ are the respective coersions for $M_G(t)(u), A(u)$, and M_u^t to the type $P \rightarrow [0,1]$; where P is the direct product of every P_u.

Demonstration *Let be the universe U of a linguistic variable $\mathcal{L}$. Given the associated family of type-1 fuzzy sets:*

$$\gamma(u) = \{M_G(t)(u), A(u), M_u^t : P_u \rightarrow [0,1]\}. \tag{10}$$

Form the product:

$$P = \prod_{u \in U} P_u. \tag{11}$$

For each $u \in U$, define the following type-1 fuzzy sets: $\overline{M_G}(t)(u), \overline{A}(u), \overline{M}_u^t : P \to [0,1]$, such that:

$$\begin{array}{l} \overline{M_G}(t)(u) = M_G(t)(u) \circ \pi_u \\ \overline{A}(u) = A(u) \circ \pi_u \\ \overline{M}_u^t = M_u^t \circ \pi_u. \end{array} \tag{12}$$

By construction given $x \in P_u$, there is $\vec{z}$, such that:

- $M_G(t)(u)(x) = \overline{M_G}(t)(u)(\vec{z})$,
- $A(u)(x) = \overline{A}(u)(\vec{z})$, *and*
- $M_u^t(x) = \overline{M_u^t}(\vec{z})$.

Moreover,

$$\begin{array}{rcl} \overline{M_u^t}(\vec{z}) & = & M_u^t(\pi_u(\vec{z})) \\ & = & [M_G(t)(u) \wedge A(u)](\pi_u(\vec{z})) \\ & = & [M_G(t)(u)(\pi_u(\vec{z})) \wedge A(u)(\pi_u(\vec{z}))] \\ & = & [\overline{M_G}(t)(u)(\vec{z}) \wedge \overline{A}(u)(\vec{z})] \\ & = & [\overline{M_G}(t)(u) \wedge \overline{A}(u)](\vec{z}) \end{array} \tag{13}$$

Q.E.D.

The equation (13) states the required coersed version of operands for the supremum expression at equation (7). This leads to the following corollary.

Theorem 4.6 (General Fuzzification Expression (GFE)) *For every linguistic variable $\mathcal{L} = \langle N, U, G, T_G, M_G \rangle$, there is a family of coersed functions $\overline{\gamma} = \{\overline{M_G}(t)(u), \overline{A}(u), \overline{M}_u^t : P \to [0,1]\}$ and the following coersed version for equation 7:*

$$T_{G_A}(t) = \bigvee_{u \in U} (\overline{M}_u^t : P \to [0,1]) \tag{14}$$

*called here **General Fuzzification Expression (GFE)**.*

Corollary 4.7 *For every system of type-n and for every linguistic variable $\mathcal{L} = \langle N, U, G, T_G, M_G \rangle$, the general fuzzification expression gives the following fuzzy set of type-2.*

$$T_{G_A} : L(G) \to (P \to [0,1]) \tag{15}$$

Demonstration *Straightforward from the previous results, specially equation 14.* ***Q.E.D.***

Observe that equation (15) gives a special case of the T_{G_A} in the definition 2.2 where all fuzzy sets denoted by $\mathbb{FZSET}$ are just of the type $P \to [0,1]$ and all operations over fuzzy sets like infimum and supremum operations become homogeneous *over that type. Observe also that $\mathbb{FZSET}$ is a supertype which contains all possible types of the form $P \to [0,1]$.*

5 Final Remarks

In this work we developed a mathematical investigation on linguistic variables and the process of fuzzyfication when it is taken into account heterogeneous linguistic variables. The most important result is equation 14 which specifies the fuzzyfication process in systems with different levels of uncertainty in the same linguistic variable.

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