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Asymptotics for Jacobi-Sobolev orthogonal polynomials associated with non-coherent pairs of measures*

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Abstract: *Inner products of the type* $\langle f,g\rangle_S = \langle f,g\rangle_{\psi_0} + \langle f',g'\rangle_{\psi_1}$, *where one of the measures* ψ_0 *or* ψ_1 *is the measure associated with the Jacobi polynomials, are usually referred to as Jacobi-Sobolev inner products. This paper deals with some asymptotic relations for the orthogonal polynomials with respect to a class of Jacobi-Sobolev inner products. The inner products are such that the associated pairs of measures* (ψ_0,ψ_1) *are not within the concept of coherent pairs of measures.*

Keywords: *Orthogonal polynomials, Sobolev orthogonal polynomials, Asymptotics*

1 Introduction

Consider the inner product defined by

$$\langle f,g\rangle_{\psi^{(\alpha,\beta)}} = \int_{-1}^{1} f(x)g(x)d\psi^{(\alpha,\beta)}(x) = \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}dx,$$

with $\alpha,\beta > -1$. Many properties of the sequence of orthogonal polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ with respect to this inner product, known as the Jacobi polynomials, are well known. Here, we assume $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ to be a sequence of monic polynomials.

For example, it is known that

$$P_{n+1}^{(\alpha,\beta)}(x) = (x-\beta_{n+1}^{(\alpha,\beta)})P_n^{(\alpha,\beta)}(x) - \alpha_{n+1}^{(\alpha,\beta)}P_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 1,$$

where $\beta_{n+1}^{(\alpha,\beta)}$, $n \geq 0$, are real numbers and

$$\alpha_{n+1}^{(\alpha,\beta)} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)} > 0, \; n \geq 1. \tag{1}$$

Moreover, the n zeros of $P_n^{(\alpha,\beta)}$ are simple and lie inside $(-1,1)$. For more details about these polynomials see, for example, [4, 11].

Assume $|\kappa| \geq 1$, $\kappa_2 \geq 0$, $\kappa_3 \geq 0$ and $\kappa_1 \geq -\dfrac{|\kappa|}{1+|\kappa|}\,\kappa_2$. We consider the class of Sobolev inner products $\langle f,g\rangle_S$ defined as follows

$$\langle f,g\rangle_S = \langle f,g\rangle_{\psi^{(\alpha,\beta)}} + \kappa_1\langle f',g'\rangle_{\psi^{(\alpha+1,\beta+1)}} + \kappa_2\langle f',g'\rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}}, \tag{2}$$

Here, the measure $\psi^{(\alpha,\beta,\kappa,\kappa_3)}$ is such that

$$\langle f,g\rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}} = \int_{-1}^{1} f(x)g(x)d\psi^{(\alpha,\beta,\kappa)}(x) + \kappa_3\left[f(\kappa)g(\kappa)\right], \tag{3}$$

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with $d\psi^{(\alpha,\beta,\kappa)}(x) = \frac{\kappa}{\kappa - x}\, d\psi^{(\alpha+1,\beta+1)}(x)$. With the restrictions given before over $\kappa, \kappa_1, \kappa_2$ and κ_3 the inner product given in (2) is positive definite.

If $\kappa_1 \neq 0$, the pair of measures $\{d\psi^{(\alpha,\beta)}, \kappa_1 d\psi^{(\alpha+1,\beta+1)} + \kappa_2 d\psi^{(\alpha,\beta,\kappa,\kappa_3)}\}$ does not form a coherent pair according to Meijer's classification given in [6]. The concept of coherent pair of measures was introduced by Iserles *et al.* [5]. In Andrade, Bracciali, Mello, Pérez [1] the authors studied the behaviour of the zeros of Jacobi-Sobolev orthogonal polynomials for $\kappa_1 \neq 0$. In [2] one can find asymptotic properties for Gegenbauer–Sobolev orthogonal polynomials associated with non-coherent pairs of measures.

For $\kappa_1 = 0$ see ([7]) and ([9]) for asymptotic properties of Sobolev orthogonal polynomials for coherent pairs of Jacobi type.

If we denote the monic orthogonal polynomials with respect to the inner product given in (3) by $P_n^{(\alpha,\beta,\kappa,\kappa_3)}$, the following results are also known (see [3]).

$$P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x) = P_n^{(\alpha+1,\beta+1)}(x) + d_{n-1}(\kappa) P_{n-1}^{(\alpha+1,\beta+1)}(x), \qquad n \geq 1, \tag{4}$$

where $d_{n-1}(\kappa) = -\rho_n^{(\alpha,\beta,\kappa,\kappa_3)}/[\kappa \rho_{n-1}^{(\alpha+1,\beta+1)}]$. Here, $\rho_n^{(\alpha,\beta,\kappa,\kappa_3)} = \langle P_n^{(\alpha,\beta,\kappa,\kappa_3)}, P_n^{(\alpha,\beta,\kappa,\kappa_3)} \rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}}$ and $\rho_n^{(\alpha+1,\beta+1)} = \langle P_n^{(\alpha+1,\beta+1)}, P_n^{(\alpha+1,\beta+1)} \rangle_{\psi^{(\alpha+1,\beta+1)}}$.

It was shown in [3] that the polynomials $\mathcal{S}_n$ satisfy $\mathcal{S}_0(x) = P_0^{(\alpha,\beta)}(x) = 1$, $\mathcal{S}_1(x) = P_1^{(\alpha,\beta)}(x) = x$ and

$$\mathcal{S}_{n+1}(x) + a_n \mathcal{S}_n(x) = P_{n+1}^{(\alpha,\beta)}(x) + b_n P_n^{(\alpha,\beta)}(x), \quad n \geq 1, \tag{5}$$

where $b_n = b_n(\kappa) = d_{n-1}(\kappa)(n+1)/n$ and for $n \geq 1$,

$$a_n = a_n(\kappa, \kappa_1, \kappa_2) = \frac{\rho_n^{(\alpha,\beta)} + \kappa_1 n^2 \rho_{n-1}^{(\alpha+1,\beta+1)}}{\rho_n^{(S)}}\; b_n(q), \tag{6}$$

where $\rho_n^{(S)} = \langle \mathcal{S}_n, \mathcal{S}_n \rangle_S$. The coefficients $a_n,\ n \geq 2$, can be recursively generated by

$$a_n = \frac{\nu_n(\kappa_1) + \alpha_n{}^{(\alpha+1,\beta+1)}}{\nu_n(\kappa_1) + \alpha_n{}^{(\alpha+1,\beta+1)} + b_{n-1}\big\{ -n(n-1)\kappa_2\kappa + \nu_{n-1}(\kappa_1)[b_{n-1} - a_{n-1}]\big\}}\; b_n, \tag{7}$$

with $\quad a_1 = \dfrac{\nu_1(\kappa_1)}{\nu_1(\kappa_1) + \kappa_2\, \rho_0^{(\alpha,\beta,\kappa,\kappa_3)}/\rho_0^{(\alpha+1,\beta+1)}}\; b_1$ and $\nu_n(\kappa_1) = n^2\kappa_1 + n/(n+\alpha+\beta+1),\ n \geq 1$.

Our objective in this paper is to consider asymptotic results associated with the orthogonal polynomials $\mathcal{S}_n$ with respect to the inner product $\langle f, g \rangle_S$ in (2) when $|\kappa| \geq 1$, $\kappa_1 > 0$, $\kappa_2 \geq 0$ and $\kappa_3 \geq 0$.

2 Some preliminary asymptotic results

Using results given in [4] or [11] one can easily verify that as $n \to \infty$,

$$\frac{\rho_n^{(\alpha,\beta)}}{\rho_{n+1}^{(\alpha,\beta)}} = \frac{1}{\alpha_{n+2}^{(\alpha,\beta)}} \to 4 \qquad \text{and} \qquad \frac{\rho_{n+1}^{(\alpha,\beta)}}{\rho_n^{(\alpha+1,\beta+1)}} \to 1. \tag{8}$$

We now give the asymptotic behaviour of the coefficients $d_n(\kappa)$ as well as of the rational functions such as $P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)/P_n^{(\alpha+1,\beta+1)}(x)$ and $P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)/P_n^{(\alpha,\beta)}(x)$, in terms of the complex function Φ defined by

$$\Phi(z) = z + \sqrt{z^2 - 1}, \quad \text{for } z \in \overline{\mathbb{C}} \setminus [-1, 1].$$

The square root in Φ is such that $\sqrt{z^2-1} > 0$ when $z > 1$ and $\sqrt{z^2-1} < 0$ when $z < -1$. Here, $\mathbb{C}$ denotes the complex plane and $\overline{\mathbb{C}}$ denotes the extended complex plane.

Fist we will need some previous results.

From results given in Nevai [8] (see also Pan [9, Lemma 3.1]) we obtain the well known ratio asymptotics for the Jacobi polynomials

$$\lim_{n\to\infty} \frac{P_{n+1}^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(x)} = \frac{\Phi(x)}{2}, \tag{9}$$

$$\lim_{n\to\infty} \frac{P_n^{(\alpha+1,\beta+1)}(x)}{P_n^{(\alpha,\beta)}(x)} = \frac{\Phi(x)}{2\sqrt{x^2-1}} = \frac{\Phi'(x)}{2}, \tag{10}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$.

Now we state the following lemma given in Pan [9, Lemmas 4.3, 4.4 and 4.5].

Lemma 2.1 *Uniformly on compact subsets of* $\overline{\mathbb{C}} \setminus [-1,1]$,

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)}{P_n^{(\alpha,\beta,\kappa)}(x)} = \begin{cases} 1 - \dfrac{\Phi(x)-\Phi(\kappa)}{x-\kappa}\,\dfrac{\sqrt{\kappa^2-1}}{\Phi(x)}, & if \quad \kappa_3 \geq 0, \\ 1, & if \quad \kappa_3 = 0, \end{cases} \tag{11}$$

$$\lim_{n\to\infty} \frac{P_n^{(\alpha+1,\beta+1)}(x)}{P_n^{(\alpha,\beta,\kappa)}(x)} = \frac{\Phi(x)-\Phi(\kappa)}{2(x-\kappa)}, \tag{12}$$

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\beta,\kappa)}(x)}{P_n^{(\alpha,\beta)}(x)} = \frac{x-\kappa}{\Phi(x)-\Phi(\kappa)}\;\Phi'(x). \tag{13}$$

We will also need the two results bellow.

Lemma 2.2 (Pan [9], Thm. 4.6) *The following limit holds*

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)}{P_n^{(\alpha,\beta)}(x)} = \begin{cases} \dfrac{x-\kappa}{\Phi(x)-\Phi(\kappa)}\;\Phi'(x) - \dfrac{\sqrt{\kappa^2-1}}{\sqrt{x^2-1}}, & if \quad \kappa_3 \geq 0, \\ \dfrac{x-\kappa}{\Phi(x)-\Phi(\kappa)}\;\Phi'(x), & if \quad \kappa_3 = 0, \end{cases} \tag{14}$$

locally uniformly in $\overline{\mathbb{C}} \setminus [-1,1]$.

Lemma 2.3 (Pan [9], Thm. 4.7) *For the coefficients* $d_n(\kappa)$ *in* (4) *and also for the coefficients* $b_n(\kappa)$ *in* (5) *the following holds*

$$\lim_{n\to\infty} d_n(\kappa) = \lim_{n\to\infty} b_n(\kappa) = b(\kappa), \tag{15}$$

where $\quad 2b(\kappa) = \begin{cases} -\Phi(\kappa), & if \ \kappa_3 > 0, \\ -[\Phi(\kappa)]^{-1}, & if \kappa_3 = 0. \end{cases}$

From (4) and previous lemma we obtain

Corollary 2.1

$$\lim_{n\to\infty} \frac{\rho_n^{(\alpha,\beta,\kappa,\kappa_3)}}{\rho_{n-1}^{(\alpha+1,\beta+1)}} = \begin{cases} \dfrac{\kappa\Phi(\kappa)}{2}, & if \quad \kappa_3 > 0, \\ \dfrac{\kappa}{2\Phi(\kappa)}, & if \quad \kappa_3 = 0. \end{cases}$$

3 Asymptotic properties

The following theorem gives information for the norm $\|\mathcal{S}_n\|_S = [\rho_n^S]^{1/2}$.

Theorem 3.1

$$\gamma_n^{(\alpha,\beta,\kappa_1)} + n^2\kappa_2\rho_{n-1}^{(\lambda,q,\kappa_1,\kappa_2)} \le \rho_n^S \le \gamma_n^{(\alpha,\beta,\kappa_1)} + n^2\kappa_2\rho_{n-1}^{(\alpha,\beta,\kappa,\kappa_3)} + b_{n-1}^2(q)\gamma_{n-1}^{(\alpha,\beta,\kappa_1)},$$

where $\gamma_n^{(\alpha,\beta,\kappa_1)} = \rho_n^{(\alpha,\beta)} + \kappa_1\rho_{n-1}^{(\alpha+1,\beta+1)}, \; n \ge 1.$

Proof: Since the monic orthogonal polynomial of degree n with respect to any inner product has the smallest norm among all the monic polynomials of degree n, we have

$$\begin{aligned}\rho_n^S &= \langle \mathcal{S}_n, \mathcal{S}_n\rangle_S = \langle \mathcal{S}_n, \mathcal{S}_n\rangle_{\psi^{(\alpha,\beta)}} + \kappa_1\langle \mathcal{S}_n', \mathcal{S}_n'\rangle_{\psi^{(\alpha+1,\beta+1)}} + \kappa_2\langle \mathcal{S}_n', \mathcal{S}_n'\rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}}\\ &\ge \rho_n^{(\alpha,\beta)} + n^2\kappa_1\rho_{n-1}^{(\alpha+1,\beta+1)} + n^2\kappa_2\rho_{n-1}^{(\alpha,\beta,\kappa,\kappa_3)}.\end{aligned}$$

To prove the right side inequality, we use

$$\rho_n^S = \langle \mathcal{S}_n, \mathcal{S}_n\rangle_S \le \langle P_n^{(\alpha,\beta)} + b_{n-1}(\kappa)P_{n-1}^{(\alpha,\beta)}, P_n^{(\alpha,\beta)} + b_{n-1}(\kappa)P_{n-1}^{(\alpha,\beta)}\rangle_S.$$

■

We can now prove the following result.

Theorem 3.2 *The coefficients* $a_n = a_n(\kappa,\kappa_1,\kappa_2)$ *in* (5) *satisfy*

$$\lim_{n\to\infty} a_n(\kappa,\kappa_1,\kappa_2) = a(\kappa,\kappa_1,\kappa_2) = -\frac{1}{2\Phi(\tilde{\kappa})}, \qquad \text{where } \tilde{\kappa} = \frac{\kappa(\kappa_1+\kappa_2)}{\kappa_1}. \tag{16}$$

Proof: From previous theorem we obtain

$$b_n[b_n - a_n] \ge 0 \quad \text{and} \quad |a_n| \le \frac{\gamma_n^{(\alpha,\beta,\kappa_1)}}{\gamma_n^{(\alpha,\beta,\kappa_1)} + n^2\kappa_2\rho_{n-1}^{(\alpha,\beta,\kappa,\kappa_3)}}|b_n|.$$

If $a = \lim_{n\to\infty} a_n$ exists, then from the inequality in Theorem 3.1, Eqs. (8) and (**??**) and Lemma 2.3,

$$0 \le |a| \le \frac{\kappa_1}{\kappa_1 - 4\kappa_2\kappa b}|b|. \tag{17}$$

From (7), we obtain

$$a_n = \frac{\delta_n\ \alpha_n^{(\alpha+1,\beta+1)}}{\delta_n\ \alpha_n^{(\alpha+1,\beta+1)} + b_{n-1}\frac{n-1}{n}\left[-\kappa_2\kappa + \delta_{n-1}\frac{n-1}{n}(b_{n-1} - a_{n-1})\right]}\ b_n, \tag{18}$$

where $\delta_n = \nu_n(\kappa_1)/n^2$. From (1) note that $\alpha_n^{(\alpha+1,\beta+1)} \to 1/4$ and $\delta_n \to \kappa_1$ as $n \to \infty$.

Thus, if $a = \lim_{n\to\infty} a_n$ exists, then from (18),

$$a^2 - \left[\frac{1}{4b(\kappa)} + b(\kappa) - \frac{\kappa_2\kappa}{\kappa_1}\right]a + \frac{1}{4} = 0. \tag{19}$$

Both possibilities for $b(\kappa)$, given in Lemma 2.3, lead to

$$a^2 + \kappa\left(1 + \frac{\kappa_2}{\kappa_1}\right)a + \frac{1}{4} = 0.$$

Hence, choosing the solution that satisfies the restriction given by (17), we obtain

$$a = -\frac{1}{2}\left[\Phi\left(\frac{\kappa(\kappa_1+\kappa_2)}{\kappa_1}\right)\right]^{-1}.$$

We now confirm that $\lim_{n\to\infty} a_n = a$ as given before. From (18), we obtain

$$\begin{aligned}
|a_n - a| \;\leq\; & \left|\frac{\delta_n\ \alpha_n^{(\alpha+1,\beta+1)}(b_n-a) + [\kappa_2\kappa - \delta_{n-1}b_{n-1} + \delta_{n-1}]\frac{n-1}{n}b_{n-1}a}{\delta_n\ \alpha_n^{(\alpha+1,\beta+1)} + b_{n-1}\frac{n-1}{n}\left[-\kappa_2\kappa + \delta_{n-1}\frac{n-1}{n}(b_{n-1}-a_{n-1})\right]}\right| \\
& + \left|\frac{\delta_{n-1}b_{n-1}a(a_{n-1}-a)}{\delta_n\ \alpha_n^{(\alpha+1,\beta+1)} + b_{n-1}\frac{n-1}{n}\left[-\kappa_2\kappa + \delta_{n-1}\frac{n-1}{n}(b_{n-1}-a_{n-1})\right]}\right| \\
\leq\; & \left|\frac{\delta_n\ \alpha_n^{(\alpha+1,\beta+1)}(b_n-a) + [\kappa_2\kappa - \delta_{n-1}b_{n-1} + \delta_{n-1}]\frac{n-1}{n}b_{n-1}a}{\delta_n\ \alpha_n^{(\alpha+1,\beta+1)} - \frac{n-1}{n}\kappa_2\kappa b_{n-1}}\right| \\
& + \left|\frac{\delta_{n-1}b_{n-1}a}{\delta_n\ \alpha_n^{(\alpha+1,\beta+1)} - \frac{n-1}{n}\kappa_2\kappa b_{n-1}}\right| |a_{n-1}-a|.
\end{aligned}$$

The latter part above is a consequence of $b_n[b_n - a_n] \geq 0$. From (19),

$$\begin{aligned}
\lim_{n\to\infty}&\left[\delta_n\ \alpha_n^{(\alpha+1,\beta+1)}(b_n-a) + (\kappa_2\kappa - \delta_{n-1}b_{n-1} + \delta_{n-1})\frac{n-1}{n}b_{n-1}a\right] \\
&= \kappa_1 b\left[a^2 - \left(\frac{1}{4b} - \frac{\kappa_2\kappa}{\kappa_1} + b\right)a + \frac{1}{4}\right] = 0.
\end{aligned}$$

Therefore,

$$\limsup|a_n - a| \leq \left|\frac{\kappa_1}{\kappa_1 - 4b\kappa_2\kappa}\right| |4ba| \limsup|a_{n-1}-a|.$$

Thus, the convergence of a_n is established if we prove that $\left|\frac{\kappa_1}{\kappa_1 - 4b\kappa_2\kappa}\right| |4ba| < 1$. Clearly, $\left|\frac{\kappa_1}{\kappa_1 - 4b\kappa_2\kappa}\right| < 1$ with the assumptions $\kappa_1 > 0$. Now,

$$|4ab| = \begin{cases} \left|\dfrac{\Phi(\kappa)}{\Phi(\tilde{\kappa})}\right|, & if \quad \kappa_3 > 0, \\ \left|\dfrac{1}{4\Phi(\kappa)\Phi(\tilde{\kappa})}\right|, & if \quad \kappa_3 = 0. \end{cases}$$

Since $|\kappa| \geq 1$ and $|\Phi(\tilde{\kappa})| > |\Phi(\kappa)| \geq 1$ then $|4ab| < 1$. Thus, the theorem is proved. ∎

Using that $a_n = b_n(\kappa)[\rho_n^{(\alpha,\beta)} + n^2\kappa_1\rho_{n-1}^{(\alpha+1,\beta+1)}]/\rho_n^S$ and $b_n(\kappa) = -[(n-1)\rho_n^{(\alpha,\beta,\kappa,\kappa_3)}]/[n\kappa\rho_{n-1}^{(\alpha+1,\beta+1)}]$, together with the results in (8), Lemma 2.3 and Theorem 3.2, we obtain the following limit

$$\lim_{n\to\infty}\frac{n^2\rho_n^{(\alpha,\beta,\kappa,\kappa_3)}}{\rho_n^S} = \frac{\kappa}{2\kappa_1\Phi(\tilde{\kappa})}.$$

Theorem 3.3 *The Sobolev orthogonal polynomials $\mathcal{S}_n(x)$ satisfy*

$$\lim_{n\to\infty}\frac{\mathcal{S}_n(x)}{P_n^{(\alpha,\beta)}(x)} = \begin{cases} \dfrac{\Phi(x)-\Phi(\kappa)}{\Phi(x)-1/\Phi(\tilde{\kappa})}, & if \quad \kappa_3 > 0, \\ \dfrac{\Phi(x)-1/\Phi(\kappa)}{\Phi(x)-1/\Phi(\tilde{\kappa})}, & if \quad \kappa_3 = 0, \end{cases} \tag{20}$$

$$\lim_{n\to\infty}\frac{\mathcal{S}_{n+1}(x)}{\mathcal{S}_n(x)} = \frac{\Phi(x)}{2}, \tag{21}$$

uniformly on compact subsets of $\overline{\mathbb{C}}\setminus[-1,1]$.

Proof: From the recurrence relation (5), we can write

$$f_{n+1}(x) = 1 + g_n(x) + h_n(x) f_n(x), \tag{22}$$

with the analytic functions on $\mathbb{C} \setminus [-1,1]$

$$f_n(x) = \frac{\mathcal{S}_n(x)}{P_n^{(\alpha,\beta)}(x)}, \quad g_n(x) = b_n(\kappa)\frac{P_n^{(\alpha,\beta)}(x)}{P_{n+1}^{(\alpha,\beta)}(x)} \text{ and } h_n(x) = -a_n\frac{P_n^{(\alpha,\beta)}(x)}{P_{n+1}^{(\alpha,\beta)}(x)}.$$

Then equations (15), (9) and (16) give

$$\lim_{n\to\infty} g_n(x) = g(x) = \frac{2b(\kappa)}{\Phi(x)} \quad \text{and} \quad \lim_{n\to\infty} h_n(x) = h(x) = -\frac{2a}{\Phi(x)}, \tag{23}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$.

Note that $|h(x)| < 1$ and that $|g(x)|$ is also bounded for all $x \in \overline{\mathbb{C}} \setminus [-1,1]$. Hence there exist positive constants $A < 1$ and B such that for all $n \geq N$

$$|h_n(x)| \leq A < 1 \quad \text{and} \quad |g_n(x)| \leq B \quad \text{if} \quad x \in \overline{\mathbb{C}} \setminus [-1,1].$$

Hence from (22),

$$\begin{aligned}
|f_{N+1}(x)| &\leq 1 + B + A|f_N(x)|, \\
|f_{N+2}(x)| &\leq 1 + B + A(1+B) + A^2|f_N(x)|, \\
\vdots & \qquad \vdots \\
|f_{N+i}(x)| &\leq \frac{(1+B)(1-A^i)}{1-A} + A^i|f_N(x)| < \frac{1+B}{1-A} + |f_N(x)|.
\end{aligned}$$

Therefore, f_n is uniformly bounded on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$.

We now show that $\{f_n\}$ converges uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$. If the limit $f(x)$ exists, then from (22) it satisfies

$$f(x) = 1 + g(x) + h(x)\, f(x). \tag{24}$$

Thus, from (23),

$$f(x) = \frac{\Phi(x) + 2b}{\Phi(x) + 2a}. \tag{25}$$

From (22) and (24),

$$|f_{n+1}(x) - f(x)| \leq |g_n(x) - g(x)| + |h_n(x) - h(x)|\,|f_n(x)| + |h(x)|\,|f_n(x) - f(x)|.$$

Since f_n is bounded for all $x \in \overline{\mathbb{C}} \setminus [-1,1]$,

$$\limsup |f_{n+1}(x) - f(x)| \leq A \; \limsup |f_n(x) - f(x)|.$$

Consequently, the convergence of $f_n(x)$ to $f(x)$ follows from $0 < A < 1$.

Using the results of Lemma 2.3 and Theorem 3.2, from (25) we obtain the limit in (20).

Since

$$\frac{\mathcal{S}_{n+1}}{\mathcal{S}_n(x)} = \frac{\mathcal{S}_{n+1}(x)}{P_{n+1}^{(\alpha,\beta)}(x)} \, \frac{P_{n+1}^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(x)} \frac{P_n^{(\alpha,\beta)}(x)}{\mathcal{S}_n(x)},$$

from (20) and (9), the limit result in (21) immediately follows. ■

Now, as we can write

$$\frac{S_n(x)}{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)} = \frac{S_n(x)}{P_n^{(\alpha,\beta}(x)} \, \frac{P_n^{(\alpha,\beta}(x)}{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)},$$

from (20) and (14) we obtain the following result.

Corollary 3.1 *Uniformly on compact subsets of* $\overline{\mathbb{C}} \setminus [-1,1]$,

$$\lim_{n\to\infty} \frac{S_n(x)}{P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x)} = \begin{cases} \dfrac{\Phi(x)-\Phi(\kappa)}{\Phi(x)-1/\Phi(\tilde{\kappa})} \left[\dfrac{x-\kappa}{\Phi(x)-\Phi(\kappa)}\, \Phi'(x) - \dfrac{\sqrt{\kappa^2-1}}{\sqrt{x^2-1}} \right]^{-1}, & if \quad \kappa_3 > 0, \\ \dfrac{\Phi(x)-\Phi(\kappa)}{\Phi(x)-1/\Phi(\tilde{\kappa})}\, \dfrac{x-\kappa}{\Phi(x)-\Phi(\kappa)}\, \Phi'(x), & if \quad \kappa_3 > 0. \end{cases}$$

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