The Probability Density Function to the Random Linear Transport Equation

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Abstract: *We present a formula to calculate the probability density function of the solution of the random linear transport equation in terms of the density functions of the velocity and the initial condition. We also present an expression for the joint probability density function of the solution in two different points. Our results have shown good agreement with Monte Carlo simulations.*
Key words: random linear transport equation, probability density function, random velocity.

1 Introduction

Uncertainties in data are natural in many models of real world problems that use partial differential equations, specially in physical sciences. However, the deterministic formulations have been traditional and convenient. Parameters of differential equations are, in general, viewed as well defined local quantities that can be assigned at each point. In practice such parameters can, at best, be measure at selected locations and interpretative procedures are used at points where measurements are not available. Quite often the support of measurements is uncertain and the data are corrupted by experimental and interpretative errors. These errors and uncertainties render the random parameters and, thus, the corresponding stochastic differential equations.

Once the statistical properties of relevant random parameters have been inferred from data, the next step is to solve random differential equations. In this paper we deal with the random linear transport equation,

$$\begin{aligned} &Q_t + A\,Q_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ &Q(x,0) = Q_0(x), \end{aligned} \tag{1}$$

where the velocity, A, is a random variable and the initial condition, $Q_0(x)$, is a random function.

The Monte Carlo method [2] has been widely used. This procedure entails generating numerous equally likely random realization of the parameter fields, solving the deterministic differential equation

$$\begin{aligned} &q_t + a\,q_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ &q(x,0) = q_0(x), \end{aligned} \tag{2}$$

for each realization, and averaging the results to obtain the statistical moments. The solution of (2) is given by $q(x,t) = q_0(x - at)$. In general, numerical methods are used to solve the deterministic realization. The Monte Carlo method has the advantage of applying to a very broad range of both linear and nonlinear problems with complex geometry and boundary condition. However, numerical errors in solving the deterministic equations, statistical errors in generating the realizations and the large number of realizations required make this method impracticable in complex situations in two-dimensional or three-dimensional problems.

The second approach includes the methods by which one seeks the statistical moments of the solution. In this approach, the main effort is usually concentrated [4, 8, 9] on the derivation of appropriate differential equations for average quantities using, in general, small perturbations with some kind of closure.

In recent years polynomial chaos has been used as a complete basis to represent random processes in stochastic Galerkin projections, transforming the stochastic equation in a set of deterministic equations (see [3, 5], for instance, and the references therein).

Although the complete solution of a random differential equation would be the (joint) Probability Density Function (PDF), it is difficult to construct differential equations for this function. In a PDF method the density function is modeled, in one point and one time, by evolution equations. As far as we know, for only a few fields have PDF methods been developed . For instance, much progress has been made in studying turbulent flows [7].

In this paper, we present a semi-explicit expression for the PDF of the solution of (1). The same ideas are also used to obtain the two-point joint PDF in terms of the velocity and initial condition distributions. In Section 4 our approach is compared with Monte Carlo simulations.

2 The Probability Density Function

Let U be a random variable with PDF given by f_U. Its cumulative distribution function is

$$F_U(u) = \mathcal{P}(U \le u) = \int_{-\infty}^{u} f_U(s) ds.$$

We begin with the random Riemann problem, a particular case of (1) in which the initial condition is given by

$$Q_0(x) = \begin{cases} L, & x < 0, \\ R, & x > 0, \end{cases} \tag{3}$$

with L and R being random variables. As shown in [1], under the natural hypothesis of independence between the velocity and the initial condition, the solution of (3) is

$$Q(x,t) = L + [R - L]\, B(x/t), \tag{4}$$

where B is the Bernoulli random variable with $\mathcal{P}(B(\xi) = 1) = F_A(\xi)$, where $\mathcal{P}$ denotes the probability measure. Following the same ideas we may show that, from (4),

$$\begin{aligned} F_{Q(x,t)}(q) &\equiv F_Q(q;x,t) = \mathcal{P}(Q(x,t) \le q) = \mathcal{P}(Q(x,t) \le q \mid B(x/t) = 0)\, \mathcal{P}(B(x/t) = 0) \\ &+ \mathcal{P}(Q(x,t) \le q \mid B(x/t) = 1)\, \mathcal{P}(B(x/t) = 1) = \mathcal{P}(L \le q)\, [1 - F_A(x/t)] + \mathcal{P}(R \le q)\, F_A(x/t) \\ &= F_L(q) + [F_R(q) - F_L(q)]\, F_A(x/t), \end{aligned} \tag{5}$$

where $\mathcal{P}(U \mid V)$ denotes the conditional probability of U given V, and F_L and F_R are the cumulative distribution functions of the initial states. Similarly, from (3),

$$F_{Q_0(x)}(q) \equiv F_{Q_0}(q;x) = F_L(q) + [F_R(q) - F_L(q)]\, H(x), \tag{6}$$

where $H(x)$ is the Heaviside (unitary) step function, i.e.,

$$H(x) = \begin{cases} 1, & \text{if} \quad x \ge 0, \\ 0, & \text{if} \quad x < 0. \end{cases} \tag{7}$$

Using the definition of $F_A(x/t)$ in (5), we have

$$\begin{aligned} F_Q(q;x,t) &= F_L(q) + [F_R(q) - F_L(q)] \int_{-\infty}^{x/t} f_A(a)\, da \\ &= F_L(q) + [F_R(q) - F_L(q)] \int_{-\infty}^{\infty} f_A(a)\, H(x - at)\, da \\ &= \int_{-\infty}^{\infty} f_A(a)\, \{F_L(q) + [F_R(q) - F_L(q)]\, H(x - at)\}\, da = \int_{-\infty}^{\infty} f_A(a)\, F_{Q_0}(q; x - at)\, da. \end{aligned} \tag{8}$$

Therefore, the differentiation with respect to q gives the PDF,

$$f_Q(q;x,t) = \int_{-\infty}^{\infty} f_A(a)\, f_{Q_0}(q;x-at)\, da = \mathrm{E}_A[f_{Q_0}(q;x-At)], \tag{9}$$

where E_A denotes the expected value relative to the random variable A. All in all, the solution given by equation (4) for the random Riemann problem leads us to the expression of the PDF of the solution at a fixed (x,t): equation (9) is mathematically attractive and stimulated us in showing that it holds for the general initial condition case.

Proposition 2.1. *The PDF of the solution of (1), at a fixed (x,t), $f_Q(q;x,t)$, is given by (9).*

Proof 1. The concept of conditional probability will play a role in calculating the cumulative probability function of $Q(x,t)$, $F_Q(q;x,t)$, for a fixed (x,t). In fact, by the *Law of Total Probability* [6] we can write

$$F_Q(q;x,t) = \mathcal{P}(Q(x,t) \le q) = \mathrm{E}_{X_0}[\mathcal{P}(Q(x,t) \le q \mid X_0)], \tag{10}$$

where X_0 is the random function given by $X_0(x,t) = x - At$. By the characteristic method we observe that $Q(x,t) \le q$ given that $X_0 = x_0$ is equivalent to $Q_0(x_0) \le q$. Thus,

$$F_Q(q;x,t) = \int_{-\infty}^{\infty} \mathcal{P}(Q_0(x_0) \le q)\, f_{X_0}(x_0)\, dx_0 = \int_{-\infty}^{\infty} F_{Q_0}(q;x_0)\, f_{X_0}(x_0)\, dx_0, \tag{11}$$

where $F_{Q_0}(q;x_0)$ is the cumulative probability function of $Q_0(x_0)$ given in (1). Taking the derivative with respect to q, we have

$$f_Q(q;x,t) = \int_{-\infty}^{\infty} f_{Q_0}(q;x_0)\, f_{X_0}(x_0)\, dx_0. \tag{12}$$

To find the PDF for X_0 we use the fact that

$$F_{X_0}(x_0) = \mathcal{P}(x - At \le x_0) = \mathcal{P}(A \ge (x-x_0)/t) = 1 - F_A((x-x_0)/t). \tag{13}$$

Differentiating (13) with respect to x_0 we arrive at

$$f_{X_0}(x_0) = \frac{d}{dx_0} F_{X_0}(x_0) = \frac{1}{t} f_A((x-x_0)/t). \tag{14}$$

Then, substituting (14) in (12), we obtain

$$f_Q(q;x,t) = \int_{-\infty}^{\infty} \frac{1}{t} f_A((x-x_0)/t)\, f_{Q_0}(q;x_0)\, dx_0 = \int_{-\infty}^{\infty} f_A(a)\, f_{Q_0}(q;x-at)\, da, \tag{15}$$

and the result follows. □

From Proposition (2.1) it follows that the m-th moment, $\mu^m(x,t)$, $m \ge 1$, of the solution of (1) is given by

$$\begin{aligned} \mu^m(x,t) &= \int_{-\infty}^{\infty} q^m\, f_Q(q;x,t)\, dq = \int_{-\infty}^{\infty} q^m \int_{-\infty}^{\infty} f_A(a)\, f_{Q_0}(q;x-at)\, da\, dq \\ &= \int_{-\infty}^{\infty} f_A(a) \int_{-\infty}^{\infty} q^m\, f_{Q_0}(q;x-at)\, dq\, da = \int_{-\infty}^{\infty} f_A(a)\, \mu_0^m(x-at)\, da, \end{aligned} \tag{16}$$

where $\mu_0^m(x)$ is the m-th moment of $Q_0(x)$. Therefore,

$$\mu^m(x,t) = \mathrm{E}_A[\mu_0^m(x-At)]. \tag{17}$$

3 The Joint Probability Density Function

Let $Q_1 = Q(x,t)$ and $Q_2 = Q(y,\tau)$ be the random solutions to (1) at (x,t) and (y,τ) ($t,\tau > 0$), respectively. In this section we study the joint cumulative distribution of Q_1 and Q_2, $F_Q(q_1,q_2;x,t,y,\tau)$, defined by

$$F_Q(q_1,q_2;x,t,y,\tau) = \mathcal{P}(Q_1 \le q_1,\ Q_2 \le q_2) = \mathrm{E}_{X_0,Y_0}[\mathcal{P}(Q_1 \le q_1,\ Q_2 \le q_2 \mid X_0,\ Y_0)], \qquad (18)$$

where, again, we have used the *Law of Total Probability* [6]. Here, X_0 and Y_0 are the random functions $X_0(x,t) = x - At$ and $Y_0(y,\tau) = y - A\tau$. For simplicity, we use for the joint distribution, F_Q, the same notation as in the previous section; the difference in the number of arguments of each one makes the context clear. The same notation will be used for the joint distribution related to Q_0. Following the ideas in Proposition 2.1, by the characteristic method, $Q_1 \le q_1$ and $Q_2 \le q_2$ given that $X_0 = x_0$ and $Y_0 = y_0$ is equivalent to $Q_0(x_0) \le q_1$ and $Q_0(y_0) \le q_2$, where Q_0 is the initial condition in (1). Therefore, from (18)

$$\begin{aligned} F_Q(q_1,q_2;x,t,y,\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{P}(Q_0(x_0) \le q_1,\ Q_0(y_0) \le q_2)\, f_{X_0,Y_0}(x_0,y_0)\, dx_0 dy_0 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{Q_0}(q_1,q_2;x_0,y_0)\, f_{X_0,Y_0}(x_0,y_0)\, dx_0 dy_0, \end{aligned} \qquad (19)$$

where $f_{X_0,Y_0}(x_0,y_0)$ is the joint PDF of X_0 and Y_0, and $F_{Q_0}(q_1,q_2;x_0,y_0)$ is the joint cumulative probability function of $Q_0(x_0)$ and $Q_0(y_0)$, which is assumed to be known. Taking the second-order mixed derivative in the above equation, we arrive at

$$f_Q(q_1,q_2;x,t,y,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Q_0}(q_1,q_2;x_0,y_0)\, f_{X_0,Y_0}(x_0,y_0)\, dx_0 dy_0. \qquad (20)$$

To determine the joint PDF, f_{X_0,Y_0}, we start with

$$\begin{aligned} F_{X_0,Y_0}(x_0,y_0) &= \mathcal{P}(X_0 \le x_0,\ Y_0 \le y_0) = \mathcal{P}(x - At \le x_0,\ y - A\tau \le y_0) \\ &= \mathcal{P}\left(A \ge (x-x_0)/t,\ A \ge (y-y_0)/\tau\right) = \mathcal{P}(A \ge m) = 1 - F_A(m), \end{aligned} \qquad (21)$$

where $m = \max\{(x-x_0)t,\ (y-y_0)/\tau\} = \max\{u,v\}$, with $u = (x-x_0)/t$ and $v = (y-y_0)/\tau$. In the sense of distributions we have that

$$\frac{\partial m}{\partial u} = H(u-v), \quad \text{and} \quad \frac{\partial m}{\partial v} = H(v-u), \qquad (22)$$

where H is given by (7). Moreover, the derivative of H is the Dirac (delta) distribution, i.e., $H'(\alpha) = \delta(\alpha)$. Consequently,

$$\begin{aligned} &\frac{\partial m}{\partial x_0} = \frac{\partial m}{\partial u}\cdot\frac{\partial u}{\partial x_0} = -\frac{1}{t}H(u-v), \quad \frac{\partial m}{\partial y_0} = \frac{\partial m}{\partial v}\cdot\frac{\partial v}{\partial y_0} = -\frac{1}{\tau}[1 - H(u-v)], \text{ and} \\ &\frac{\partial^2 m}{\partial y_0 \partial x_0} = -\frac{1}{t\tau}\delta(u-v). \end{aligned} \qquad (23)$$

Finally, taking the second-order mixed derivative in (21), we find

$$f_{X_0,Y_0}(x_0,y_0) = -\frac{\partial^2 F_A(m)}{\partial y_0 \partial x_0} = \frac{1}{t\tau} f_A(m)\delta(u-v) = \frac{1}{t\tau} f_A(u)\delta(u-v), \qquad (24)$$

since $g(\alpha)\delta(\alpha) = g(0)\delta(\alpha)$. Substituting this expression in (20) we obtain

$$\begin{aligned} f_Q(q_1,q_2;x,t,y,\tau) &= \frac{1}{t\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Q_0}(q_1,q_2;x_0,y_0)\, f_A(u)\, \delta(u-v)\, dx_0 dy_0 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Q_0}(q_1,q_2;x-at,y-b\tau)\, f_A(a)\, \delta(a-b)\, da\, db \\ &= \int_{-\infty}^{\infty} f_{Q_0}(q_1,q_2;x-at,y-a\tau)\, f_A(a)\, da = \mathrm{E}_A[f_{Q_0}(q_1,q_2;x-At,y-A\tau)]. \end{aligned} \qquad (25)$$

These arguments show the following result:

Proposition 3.1. *Let $Q_1 = Q(x,t)$ and $Q_2 = Q(y,\tau)$ be the random solution of (1) at (x,t) and (y,τ) $(t,\tau > 0)$, respectively. The joint PDF of these random variables is given by*

$$f_Q(q_1, q_2; x, t, y, \tau) = \mathrm{E}_A[f_{Q_0}(q_1, q_2; x - At, y - A\tau)]. \tag{26}$$

In view of Propositions (2.1) and (3.1), it follows that the covariance of the solution of (1) at (x,t) and (y,τ) $(t,\tau > 0)$ is given by

$$\begin{aligned}\mathrm{Cov}[Q(x,t), Q(y,\tau)] &= \mathrm{E}[Q(x,t)\, Q(y,\tau)] - \mathrm{E}[Q(x,t)]\, \mathrm{E}[Q(y,\tau)] \\ &= \mathrm{E}_A[\mathrm{Cov}(Q_0(x - At),\, Q_0(y - A\tau))] - \mathrm{E}_A[\mu_0(x - At)]\, \mathrm{E}_A[\mu_0(y - A\tau)].\end{aligned} \tag{27}$$

4 Numerical Experiments

Several computational experiments validate our approach, and show the influence of the velocity averaging process in transport differential equations. Some numerical results are presented in this section. We have also considered a simplified version of (1), taking $\mathrm{E}[A]$ as the velocity. In this case, the random initial condition is transported along the characteristic $x = x_0 + \mathrm{E}[A]\, t$. In our experiments, we used this simplified version to show the influence of considering the variability of random velocities in transport equations.

Example 1: The main purpose of this example is to validate our approach with the Monte Carlo method. In the computations the random initial condition, $Q_0(x)$, is normally distributed with $\mathrm{E}[Q_0(x)] = 0.5\exp(-10x^2)$ and an exponentially decaying covariance function, $\mathrm{Cov}_{Q_0}(x,y) = \sigma_0^2 \exp(-|x-y|/\beta)$. The covariance function is parameterized by the variance, $\mathrm{Var}[Q_0(x)] = \sigma_0^2$ (which is assumed to be constant), and by the correlation length, $\beta > 0$, which governs the decay rate of the time correlation. In the tests we used $\sigma_0 = 0.4$ and $\beta = 2$. In Figures 1–3 we depict the results of using expression (15) to calculate $f_Q(q; x, 1)$ with different velocity distributions, and compare them with Monte Carlo simulations. As is well known, the Monte Carlo method is based on the relative frequency approach of probability. Using the exact solution of (2) on 100 000 realizations, we computed the PDF at point $(x, 1)$. The characteristics crossing $(x, 1)$ emanate from $X_0 = x - A$ and, following our approach, $f_Q(q; x, 1)$ is the expectation of $f_{Q_0}(q; x - A)$, *i.e.*, $\mathrm{E}_A[f_{Q_0}(q; x - A)]$. As pointed out below, we also plot the distribution $f_{Q_0}(q; x - \mathrm{E}[A])$ in each experiment. We chose two representative values of x for each velocity distribution. With this information the figures show the effect of averaging the initial conditions caused by considering the velocity A as a random variable. In Figure 1 the velocity has a normal distribution, $A \sim N(0.1, 0.4)$; in Figure 2 the velocity is a log-normal random variable, $A = \exp(\xi)$, with $\xi \sim N(0.1, 0.3)$; and Figure 3 corresponds to a velocity with uniform distribution in the interval $[-0.25, 0.25]$. From the experiments we also observe that the shape of the PDF's, f_Q, is strongly dominated by the initial condition distribution, f_{Q_0}, which, in our case, is a normal distribution.

Example 2: In this example we take other PDF's for the velocity and initial condition. Figures 4 and 5 show the PDF's for a Cauchy distribution of velocity A,

$$f_A(a) = \exp\{-2|a - \mu|\}, \quad \mu = \mathrm{E}[A] = 1,$$

and two different distributions for the initial condition Q_0. In Figure 4 we depict the results for Q_0 with the same Cauchy distribution as A, but with mean $\mu(x) = |x|/2$. The expectations $m^1(x)$ for $f_{Q_0}(q; x - \mathrm{E}[A])$, and $\mu^1(x)$ for $f_Q(q; x, 1)$ are $m^1(0) = 0.50$, $\mu^1(0) = 0.53$, $m^1(1) = 0.00$ and $\mu^1(1) = 0.25$. Figure 5 shows the results for a uniform distribution of Q_0 in the interval $[\mu(x) - 1, \mu(x) + 1]$ with the same μ as in the previous case. The expectations are $m^1(-2) = 1.54$, $\mu^1(-2) = 1.50$, $m^1(1) = 0.00$ and $\mu^1(1) = 0.25$.

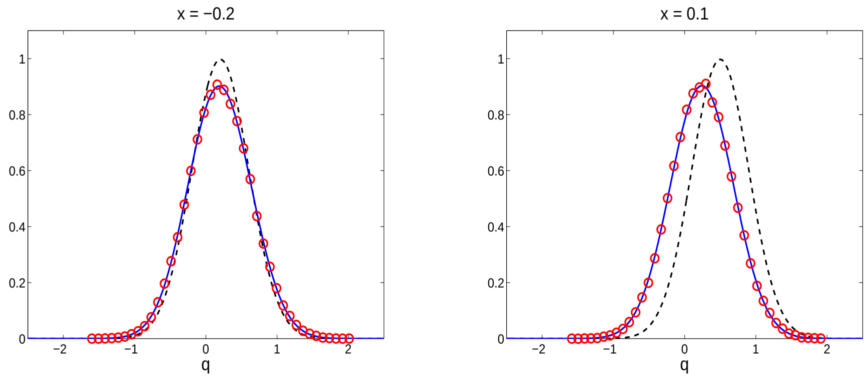

Figure 1: PDF's of the solution for a normal distribution of velocity A and a normal distribution of the initial condition: exact $f_Q(q;x,1)$ (solid line), simulated $f_Q(q;x,1)$ (circles), and averaged $f_{Q_0}(q;x-\mathrm{E}[A])$ (dashed line).

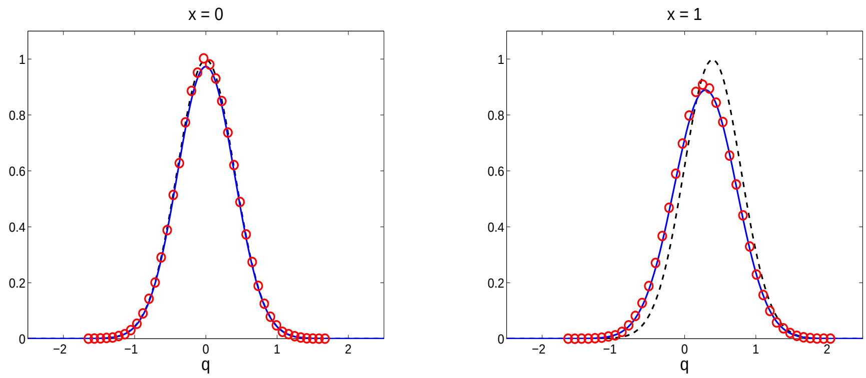

Figure 2: PDF's of the solution for a log-normal distribution of velocity A and a normal distribution of the initial condition: exact $f_Q(q;x,1)$ (solid line), simulated $f_Q(q;x,1)$ (circles), and averaged $f_{Q_0}(q;x-\mathrm{E}[A])$ (dashed line).

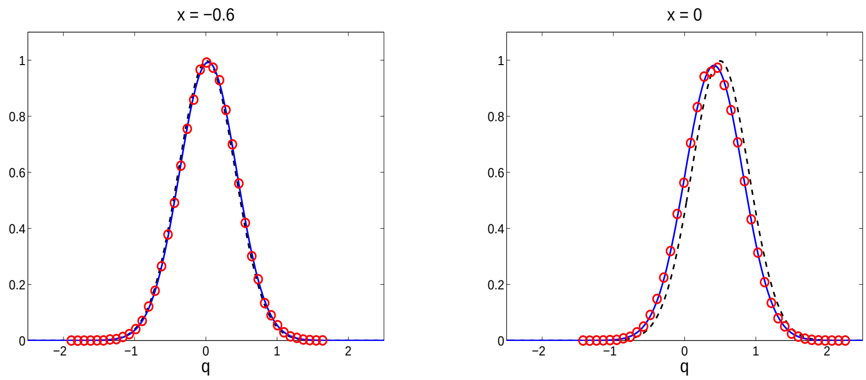

Figure 3: PDF's of the solution for a uniform distribution of velocity A and a normal distribution of the initial condition: exact $f_Q(q;x,1)$ (solid line), simulated $f_Q(q;x,1)$ (circles), and averaged $f_{Q_0}(q;x-\mathrm{E}[A])$ (dashed line).

5 Conclusion

New formulas (15), (17), (26), and (27) were deduced from basic concepts of probability theory and the characteristic method for differential equations. With numerical integration of the data distributions they are a very simple, as well as stable, way to calculate the usual measures of statistical properties of the solution of (1), the moments.

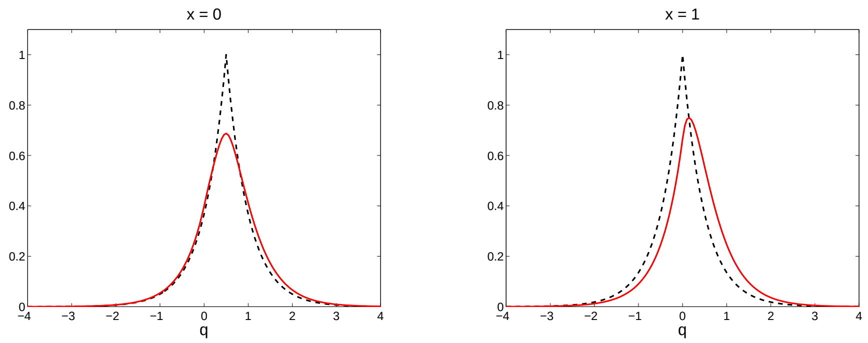

Figure 4: PDF's of the solution for a Cauchy distribution of velocity A and a Cauchy distribution of the initial condition: exact $f_Q(q;x,1)$ (solid line) and averaged $f_{Q_0}(q;x-\mathrm{E}[A])$ (dashed line).

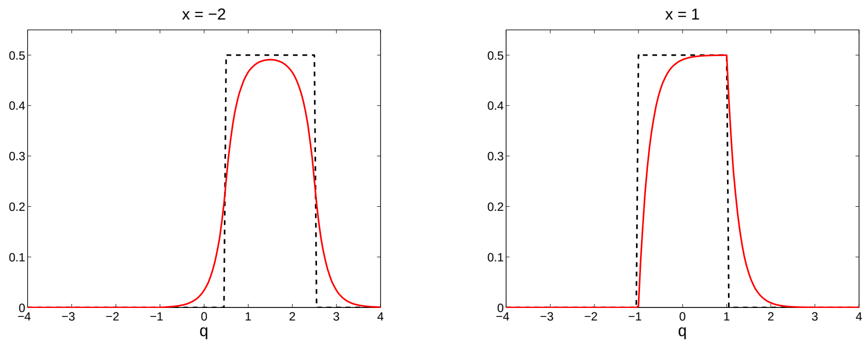

Figure 5: PDF's of the solution for a Cauchy distribution of velocity A and a uniform distribution of the initial condition: exact $f_Q(q;x,1)$ (solid line) and averaged $f_{Q_0}(q;x-\mathrm{E}[A])$ (dashed line).

References

[1] M. C. C. Cunha, F. A. Dorini, A note on the Riemann problem for random transport equation. *Computational and Applied Mathematics* 26(3) (2007) 323–335.

[2] G. S. Fishman, "Monte Carlo: concepts, algorithms and applications", Springer-Verlag, New York, 1996.

[3] R. G. Ghanem, Ingredients for a general purpose stochastic finite element formulation. *Comput. Method. Appl. Mech. Engrg.* 168 (1999) 19–34.

[4] J. Glimm, D. Sharp, "Stochastic partial differential equations: Selected applications in continuum physics", Vol. 64 of Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs, American Mathematical Society, Providence, 1998.

[5] D. Gottlieb, D.Xiu, Galerkin method for wave equations with uncertain coefficients. *Commun. Comput. Phys.* 3(2) (2008) 505–518.

[6] A. Papoulis, "Probability, Random Variables, and Stochastic Processes", McGraw-Hill, Inc., New York, 1984.

[7] S. B. Pope, "Turbulent Flow Computations using PDF Methods. In Recent Advances in Computational Fluid Dynamics, Lecture Notes in Engineering", Springer-Verlag, 1989.

[8] M. Shvidler, K. Karasaki, Exact averaging of stochastic equations for transport in random velocity field. *Transport in Porous Media* 50 (2003) 223–241.

[9] D. Zhang, "Stochastic Methods for Flow in Porous Media - Coping with Uncertainties", Academic Press, San Diego, 2002.