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Differential equations related to the Fleming-Viot model with jumps*

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Abstract: *This short communication deals with the so-called jump-type Fleming-Viot process. It is shown that when the mutation operator satisfy certain conditions, the random measure of the process is absolutely continuous with respect to the Lebesgue measure in* $\mathbb{R}$. *This, in turn, allows us to represent the process as the solution of a stochastic partial differential equation. We extend thus the result of Konno & Shiga [9]. An application to population genetics is presented.*
Keywords: *Jump-type Fleming-Viot process, sample path property, absolute continuity, stochastic partial differential equation.*

1 Introduction

In population genetics, Fleming-Viot processes play a major role combining suitably, in a unique setting, the main evolutionary forces which change the allelic gene frequencies in a population such as, *mutation*, *natural selection* and *random genetic drift* (see, e.g., [6, 3, 5]). If S is the set of allelic genes for one locus of interest in a population, then the Fleming-Viot process models the evolution of gene frequencies in time, assuming values in the state space $\mathcal{M}_1(S)$, which is the space of probability measures over S.

Along the same lines as in [9], we get a representation for this class of process by deriving a stochastic partial differential equation (SPDE) for the density of the random measure of the process. The main hindrance here is to deal with some *sample-path properties* for the jump-type Fleming-Viot process, which is tantamount to show that the state-space of the process, under some conditions, is contained in the set of absolute continuous measures with respect to the Lebesgue measure. Following Roelly-Coppoletta [12] and Konno & Shiga [9], we restrict our study to the one-dimensional type space case.

The framework of an SPDE representing a measure-valued stochastic process may be useful in giving some insights, mainly for applications, even if its formulation is more restrictive than the formulation of a martingale problem, for instance.[1] With respect to population genetics, it is notorious how many data is being collected in recent years, data that require theoretical models which could analyse them [14, 15], and the most disseminate approach (in any applied area) is still by the means of differential equations.

An outline of the paper follows. In Section 2 we recall the general definition of the jump-type Fleming-Viot process. By the means of a random time change which causes a mass change on the process, we obtain another jump-type measure-valued process in Section 3 that will aid us in proving Theorem 5.1. The establishment of the absolute continuity of the Fleming-Viot measure

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[1] Multiple modelling approaches to the same problem are not new in science, here we could think of the opposite approaches in quantum mechanics: wave mechanics and matrix mechanics; the former having an inestimable value for solving problems, while the latter being a true advance in understanding the microscopic world (John Gribbin, *Le chat de Schrödinger: physique quantique et réalité*, Flammarion, 1994.).

with respect to the Lebesgue measure on the real line is done in Section 4, from which we are able to write

$$Y(dx,t) = Y(x,t)dx. \tag{1}$$

In Section 5 we prove Theorem 5.1 that allow us to write the process as a SPDE with jumps, mainly we show the existence of a Gaussian white noise measure related to the continuous part of the martingale measure. We give an application to population genetics in Section 6.

2 Jump-type Fleming-Viot process

We start with our traditional setup. Let S be a one-dimensional metric space. Let $C(S)$ be the Banach space of continuous functions with the norm of the supremum ($\beta \in C(S), ||\beta|| = \sup_{x\in S}|\beta(x)|$). The set of Borel subsets of S will be denoted $\mathcal{B}(S)$, and the vector space of bounded measurable functions on S, $b\mathcal{B}(S)$. Let $\mathcal{M}_1(S)$ and $\mathcal{M}_R(S)$ be the space of probability measures and the space of finite Radon measures over $\mathcal{B}(S)$, respectively. For $\beta \in b\mathcal{B}(S)$ and $\mu \in \mathcal{M}_1$ (or $\mathcal{M}_R$) we denote $\langle \beta, \mu\rangle = \int_S \beta(x)\mu(dx)$. To $\mathcal{M}_1(S)$ and $\mathcal{M}_R(S)$ it is given the weak topology. Let $\mathcal{B}(\mathcal{M}_1)$ denote the set of Borel subsets of $\mathcal{M}_1(S)$. Define also $b\mathcal{B}(\mathcal{M}_1)$, the vector space of bounded measurable functions on $\mathcal{M}_1(S)$. And let $\overline{\eta} = \frac{\eta}{\langle 1,\eta\rangle}$, for $\eta \in \mathcal{M}_R(S)$.

Let $\Omega := D([0,\infty[, \mathcal{M}_1(S))$ be the set of *càdlàg*[2] functions from $[0,\infty[$ to $\mathcal{M}_1(S)$ equipped with the Skorokhod topology and $Y_t : \Omega \to \mathcal{M}_1(S)$ be the canonical process, $Y_t(\omega) = \omega(t)$. Let the σ-algebra $\mathcal{F}$ be the set of Borel subsets $\mathcal{B}(\Omega)$ and the filtration $\{\mathcal{F}_t\}$ in $\mathbb{D}$ be given by $\mathcal{F}_t := \cap_{\epsilon>0}\mathcal{F}^0_{t+\epsilon}$, where $\mathcal{F}^0_t := \sigma(\{Y_u\} : 0 \le u \le t)$, and $\mathcal{F}_\infty := \vee_{n\in\mathbb{N}}\mathcal{F}_n$.

Let $\mu \in \mathcal{M}_1(S)$, $a \ge 0$ and $\nu(du)$ be a measure in $\mathbb{R}$ such that $\int_0^\infty (u\wedge u^2)\nu(du) < \infty$. Fix also $g : [0,\infty[\to [0,\infty[$ a càdlàg function for which exists $\tau \in]0,\infty]$ such that $g(t) > 0$ for $t \in [0,\tau[$ and $g(t) = 0$ for $t \in [\tau,\infty[$. We consider a linear operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to b\mathcal{B}(S)$ which is the generator of a Feller semigroup $P_t : C(S) \to C(S)$.

The process $\{\mathcal{P}_\mu : \mu \in \mathcal{M}_1(S)\} \subset \mathcal{M}_1(\Omega)$ is a **jump-type Fleming-Viot process** if:
(FV1) $\mathcal{P}_\mu[Y_0 = \mu] = 1$, $\mathcal{P}_\mu[Y_t = Y_{\tau^-},\ t \ge \tau] = 1$;
(FV2) for $\beta \in \mathcal{D}(\mathcal{L})$,

$$\langle\beta, Y_t\rangle = \langle\beta, Y_0\rangle + \int_0^t \langle\mathcal{L}\beta, Y_s\rangle\, \mathbf{I}_{[s<\tau]}ds + M^c_t(\beta) + \int_0^t\int_{\mathcal{M}_R(S)} \frac{\langle 1,\eta\rangle\,\langle\beta, \overline{\eta} - Y_{s^-}\rangle}{g(s^-) + \langle 1,\eta\rangle}\mathbf{I}_{[s<\tau]}\widetilde{N}(ds, d\eta), \tag{2}$$

is a càdlàg $\mathcal{F}_t$-semimartingale, where $\{M^c(\beta)\}_{t\ge 0}$ is a continuous martingale with predictable quadratic variation given by

$$\langle\!\langle M^c(\beta)\rangle\!\rangle_t = a\int_0^t g(s)^{-1}\int_{x\in S}\int_{y\in S}\beta(x)\beta(y)Q(Y_s; dx\times dy)\mathbf{I}_{[s<\tau]}ds \tag{3}$$

with $Q(\mu; dx \times dy) = \delta_x(dy)\mu(dx) - \mu(dy)\mu(dx)$ and $\widetilde{N}$ is a discontinuous $\mathcal{F}_t$-martingale measure corresponding to a random point process N on $\mathbb{R}_+ \times \mathcal{M}_R(S)$, such that $\widetilde{N}(t, B) = N(t, B) - \hat{N}(t, B)$, where $\hat{N}$ is the compensator of N given by

$$\hat{N}(ds, d\eta) = \left\{\int_S \left[\int_0^\infty \delta_{u\delta_x}(d\eta)\nu(du)\right] Y_s(dx)\right\} ds\,. \tag{4}$$

Theorem 2.1 (Hiraba,2000) *Let $\mu \in \mathcal{M}(S)$. There exists a unique probability $\mathcal{P}_\mu$ on Ω satisfying the conditions* **FV1** *and* **FV2** *above.*

[2] Right-continuous with left limits.

Proof: See [7].

In population genetics, we interpret this process as follows. We consider S the space of allelic genes for one locus. The frequencies of these genes in a population is modelled by $\mathcal{M}(S)$. Mutation is represented by the operator $\mathcal{L}$, whose action may be seen as leading some kind of dynamic law over S. Random genetic drift is interpreted as composed of two terms: a continuous term given by M^c which represents the loss in the population genetic diversity due to random mating; and a discontinuous term, led by $\widetilde{N}$, due to abrupt changes suffered by the population. See [6, 2] for these interpretations and also [3, 5] for other factors that can be included in the model, such as, selection and gene conversion.

3 Random time change

The random time change performed here implies in a change of the mass of the process, thus originating a new measure-valued process which will be used to construct the jumping SPDE in Section 5.

Let Z_t satisfy the following stochastic differential equation[3]

$$dZ_t = \sqrt{\frac{a}{g(t)}} Z_t dB_t + cZ_t dt, \tag{5}$$

with $Z_0 = 1$, where B_t is a Brownian motion independent of Y_t (considered in an extension of $(\Omega, \mathcal{F}, \mathcal{P}_\mu)$, if necessary), and c is a constant real number such that $c > \frac{a}{2g(t)}$, for $0 < t < \tau$. Thus Z_t is equal to:

$$Z_t = \exp\left[\int_0^t \sqrt{\frac{a}{g(s)}} dB_s + ct - \int_0^t \frac{a}{2g(s)} ds\right]. \tag{6}$$

Define now the random function $C_t : [0, \tau[\to [0, \infty[$, by $C_t = \int_0^t Z_s ds$, that may be viewed as a random time change [4] or an occupation measure [8] weighted by Z_s,[4]. This random time change inspired by that of Shiga [13] became a technique for interchange between diffusions in $\mathcal{M}_1(S)$ and $\mathcal{M}_R(S)$ [9, 16].

An $\mathcal{M}_R$-valued process X_t can be defined by

$$X_t(dx) = Z_{C_t^{-1}} Y_{C_t^{-1}}(dx). \tag{7}$$

We notice that $< 1, X_t >= Z_{C_t^{-1}}$ is positive and continuous in $t \geq 0$. A direct result from this definitions is the next lemma.

Lemma 3.1 *The random measure $Y_t(dx)$ is absolutely continuous with respect to the Lebesgue measure $\lambda(dx)$ for almost all $t > 0$ if, and only if, also is $X_t(dx)$.*

Define also $\widetilde{\mathcal{F}}_t \equiv \mathcal{F}_{C_t^{-1}}$. From relation (7), we write $\langle \beta, X_t \rangle = Z_{C_t^{-1}} \left\langle \beta, Y_{C_t^{-1}} \right\rangle$, for $\beta \in \mathcal{D}(\mathcal{L})$, to which we apply Itô's formula to obtain

$$\langle \beta, X_t \rangle - \langle \beta, X_0 \rangle = \int_{]0,t]} \frac{\langle \mathcal{L}\beta, X_{s-} \rangle}{\langle 1, X_s \rangle} ds + \widetilde{M}_t(\beta) + c \int_{]0,t]} \frac{\langle \beta, X_{s-} \rangle}{\langle 1, X_s \rangle} ds. \tag{8}$$

[3]Note that the stochastic differential equation is chosen in order that its solution Z_t fits adequately in the proof of the Theorem 5.1.

[4]For each $A \in \mathcal{F}$, the weighted amount of time that Z_t occupies A during the interval $[0, t]$ is $C_t(A) = \int_0^t Z_s(A) ds$.

where

$$\widetilde{M}_t(\beta) = \int_{]0,C_t^{-1}]} Z_s dM_s^c(\beta) + \int_{]0,C_t^{-1}]} \sqrt{\frac{a}{g(s^-)}} \langle \beta, Y_{s^-} \rangle Z_s dB_s$$
$$+ \int_{]0,C_t^{-1}]} \int_{\mathcal{M}_R(S)} Z_s \frac{\langle 1,\eta \rangle}{g(s^-) + \langle 1,\eta \rangle} \langle \beta, \overline{\eta} - Y_{s^-} \rangle \widetilde{N}(ds, d\eta) \tag{9}$$

is an $\widetilde{\mathcal{F}}_t$-martingale.

4 Absolute continuity w.r.t. Lebesgue measure

We admit the following conditions about the semigroup P_t :
Conditions :
(A1) P_t admits a continuous density $p_t(x,y)$ in $(t,x,y) \in]0,\infty[\times S \times S$ with respect to the Lebesgue measure $\lambda(dx)$ on S,
(A2) there exists $0 < b < 1$ such that for every $T > 0$, $\sup_{0\le t\le T} \sup_{x,y\in S} p_t(x,y)t^b < \infty$,
(A3) there exist constants $\delta > 0$ and $\epsilon > 0$ such that for every $T > 0$ we have $D_T > 0$ and for every $x, y \in S$ and $0 < h < 1$,

$$\int_0^t \int_S [p_{s+h}(z,x) - p_s(z,y)]^2 dz ds \le D_T(|x-y|^\delta + h^\epsilon). \tag{10}$$

And now we may state the following result about the absolute continuity of the Fleming-Viot measure. The proof relies on the expressions for the *moments* of the process and will appear elsewhere.

Theorem 4.1 *Suppose that the semigroup P_t satisfies the conditions (A1-A3). Then almost surely the random measure $Y_t(dx)$ of the one-dimensional jump-type Fleming-Viot process is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}$ for almost all $t > 0$.*

5 Jumping SPDE

With the result of the absolute continuity of Y_t in hand we may proceed to deduce a stochastic partial differential equation for the jump-type Fleming-Viot process. First, we recall the mathematical structure for an $\mathcal{S}'(S)$-valued Wiener process following [9]. The space of **Schwartz distributions** $\mathcal{S}'(S)$ is the topological dual of $\mathcal{S}(S)$, the space of rapidly decreasing C^∞-functions defined on S, equipped with the Schwartz topology. An $\mathcal{S}'(S)$-valued continuous stochastic process W_t defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ is called an $\mathcal{S}'(S)$-valued standard $\mathcal{F}_t$-Wiener process, if for every $\zeta \in \mathcal{S}(S)$, $W_t(\zeta)$ is a one-dimensional $\mathcal{F}_t$-Brownian motion with the diffusion constant $||\zeta||^2_{L^2(S)}$. The correspondent Gaussian random measure $W(ds,dx)$ on $[0,\infty[\times S$ satisfies

$$W_t(\zeta) = \int_0^t \int_S \zeta(x) W(ds,dx), \text{ for every } \zeta \in \mathcal{S}(S). \tag{11}$$

Theorem 5.1 *Suppose that the semigroup P_t satisfies the conditions (A1-A3). There exists an $\mathcal{S}'(S)$-valued standard Wiener process W_t defined on an extension of the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P}_\mu)$ such that P_μ almost surely*

$$Y_t(x) = Y_0(x) + \int_0^t \mathcal{L}^* Y_s(x) \mathbf{I}_{[s<\tau]} ds + \int_0^t \sqrt{\frac{a}{g(s)}} \sqrt{Y_s(x)} \dot{W}_s(x) \mathbf{I}_{[s<\tau]} ds$$
$$- \int_0^t \sqrt{\frac{a}{g(s)}} \int_{y\in S} Y_s(x) \sqrt{Y_s(y)} \dot{W}_s(y) dy \mathbf{I}_{[s<\tau]} ds$$
$$+ \int_0^t \int_{\mathcal{M}_R(S)} \frac{\langle 1,\eta \rangle}{g(s^-) + \langle 1,\eta \rangle} \left[\frac{\delta \overline{\eta}(x)}{\delta x} - Y_{s^-}(x) \right] \mathbf{I}_{[s<\tau]} \widetilde{N}(ds, d\eta). \tag{12}$$

for every $t \geq 0$ *and* $\beta \in \mathcal{D}(\mathcal{L})$.

Proof: We make use of the processes Z_t, X_t, the martingale measure $\widetilde{M}_t$ and their relation with Y_t presented in Section 3. Notice that by Theorem 4.1 and Lemma 3.1, the random measure $X_t(dx)$ is absolutely continuous with respect to $\lambda(dx)$. Let the martingale measure $\widetilde{M}^c(ds,dx)$ be related to $\widetilde{M}^c_t(\beta)$ *via*

$$\int_0^t \int_S \beta(x)\widetilde{M}^c(ds,dx) = \widetilde{M}^c_t(\beta). \tag{13}$$

Consider an $\mathcal{S}'(S)$-valued standard $\widetilde{\mathcal{F}}_t$-Wiener process $\widetilde{W}_t$ independent of X_t and define also another an $\mathcal{S}'(S)$-valued process $\widehat{W}_t(\beta)$ by

$$\widehat{W}_t(\beta) = \int_0^t \int_S \frac{1}{\sqrt{X_s(x)}} \mathbf{I}_{[X_s(x)\neq 0]}\beta(x)\widetilde{M}^c(ds,dx) + \int_0^t \int_S \mathbf{I}_{[X_s(x)=0]}\beta(x)\dot{\widetilde{W}}_s(ds,dx) \tag{14}$$

defined thus in order to deal with X_t equal or not to zero. We may write this last relation as

$$\int_0^t \int_S \sqrt{X_s(x)}\beta(x)\dot{\widehat{W}}_s(x)dsdx = \int_0^t \int_S \beta(x)\widetilde{M}^c(ds,dx) = \widetilde{M}^c_t(\beta). \tag{15}$$

We obtain from equation (8) and (15) the following equation for $\langle \beta, X_t\rangle$

$$\begin{aligned}\langle \beta, X_t\rangle - \langle \beta, X_0\rangle = &\int_{]0,t]} \left[\frac{\langle \mathcal{L}\beta, X_{s-}\rangle}{\langle 1, X_{s-}\rangle} + c\frac{\langle \beta, X_{s-}\rangle}{\langle 1, X_{s-}\rangle}\right] ds + \int_0^t \int_S \sqrt{X_s(x)}\beta(x)\dot{\widehat{W}}_t(x)dtdx \\ &+ \int_{]0,C_t^{-1}]}\int_{\mathcal{M}_R(S)} Z_s \frac{\langle 1,\eta\rangle}{g(s^-)+\langle 1,\eta\rangle}\langle \beta, \overline{\eta} - Y_{s-}\rangle \widetilde{N}(ds,d\eta).\end{aligned} \tag{16}$$

Now, let a process W_t be given by

$$\langle \beta, W_t\rangle = \int_0^{C_t} \frac{1}{\sqrt{\langle 1, X_s\rangle}}\sqrt{\frac{g(C_s^{-1})}{a}}\left\langle \beta, d\widehat{W}_s\right\rangle. \tag{17}$$

In such way, W_t is an $\mathcal{S}'(S)$-valued standard Wiener process and

$$\sqrt{\frac{a}{g(t)}}d\langle \beta, W_t\rangle = \sqrt{Z_t}\left\langle \beta, d\widehat{W}_{C_t}\right\rangle. \tag{18}$$

Applying Itô's formula to $\langle \beta, Y_t\rangle = \langle X_{C_t}, \beta\rangle Z_t^{-1}$ and using (18) we get

$$\begin{aligned}\langle \beta, Y_t\rangle - \langle \beta, Y_0\rangle = &-\int_{]0,t]} \langle \beta, Y_{s-}\rangle \left[\sqrt{\frac{a}{g(s)}}dB_s + cds\right] + \int_{]0,t]}\int_S \sqrt{Y_s(x)}\beta(x)\sqrt{\frac{a}{g(s)}}W_s(x)dsdx \\ &+ \int_{]0,t]}\int_{\mathcal{M}_R(S)} \frac{\langle 1,\eta\rangle\langle \beta, \overline{\eta} - Y_{s-}\rangle}{g(s^-)+\langle 1,\eta\rangle}\widetilde{N}(ds,d\eta) + \int_{]0,t]} \left[\langle \mathcal{L}\beta, Y_s\rangle + c\langle \beta, Y_s\rangle\right] ds\end{aligned} \tag{19}$$

Note then that as $Z_t = \langle 1, X_{C_t}\rangle$, and using (16) with $\beta(x) = 1$, we have, by differentiating both members,

$$dB_t = \int_S \sqrt{Y_t}\dot{W}_t dtdx, \tag{20}$$

where we applied (18) again. Thus applying this last relation to (19) we can rewrite it as

$$\begin{aligned}\langle \beta, Y_t\rangle - \langle \beta, Y_0\rangle = &-\int_{]0,t]} \sqrt{\frac{a}{g(s)}}\langle \beta, Y_{s-}\rangle \int_S \sqrt{Y_t}\dot{W}_t dtdx + \int_{]0,t]}\sqrt{\frac{a}{g(s)}}\int_S \sqrt{Y_s(x)}\beta(x)W_s(x)dsdx \\ &+ \int_{]0,t]}\int_{\mathcal{M}_R(S)} \frac{\langle 1,\eta\rangle\langle \beta, \overline{\eta} - Y_{s-}\rangle}{g(s^-)+\langle 1,\eta\rangle}\widetilde{N}(ds,d\eta) + \int_{]0,t]} \langle \mathcal{L}\beta, Y_s\rangle ds\end{aligned} \tag{21}$$

Bearing in mind that Y_t belongs to $\mathcal{S}'(S)$ we get (12).

q.e.d.

Remark 5.1 *With respect to the equation* (12)*, we note that it is a natural generalization of equation* (0.6) *of Konno & Shiga [9, p. 203], and we can obtain their equation by putting* $\nu = 0$ *and* $g(s) = a$ *for all* $0 \leq s \leq \tau$. *This shows the consistency of our generalization.*

Remark 5.2 *In the sense of Schwartz distributions, the density* $Y_t(x)$ *can be viewed as satisfying the following equation*

$$\frac{\partial Y_t(x)}{\partial t} = \mathcal{L}^* Y_t(x) + \sqrt{\frac{a}{g(t)}} \int_{y \in S} \sqrt{Y_t(y)} \left[\delta_x(y) - Y_t(x)\right] \dot{W}_t(y) dy + \int_{\mathcal{M}_R(S)} \frac{\langle 1, \eta \rangle}{g(t) + \langle 1, \eta \rangle} \left[\frac{\delta \overline{\eta}(x)}{\delta x} - Y_t(x)\right] \dot{\widetilde{N}}_t(\eta) d\eta. \tag{22}$$

This equation differs from the equations commonly found in the literature [1, 11], since the jump part $\widetilde{N}(dt, d\eta)$ *does not come from a Poisson point process.*

6 An example from population genetics

Consider $0 \leq t_0 \leq t < \tau$. Now, let the **mean density of type** x **at time** t be given by

$$m(t, x) = E[Y(t, x)]. \tag{23}$$

So, for $\beta \in D(\mathcal{L})$,

$$E[\langle \beta, Y_t \rangle] = \int_{-\infty}^{+\infty} \beta(x) m(t, x) dx. \tag{24}$$

Using (22), we can easily derive the following equation in the weak form:

$$\frac{\partial m}{\partial t} = \mathcal{L}^* m(t, x). \tag{25}$$

Let, then, $m_2(t, x_1, x_2)$ be the mean joint density of (x_1, x_2) at time t, given by

$$m_2(t, x_1, x_2) = E[Y(t, x_1) Y(t, x_2)]. \tag{26}$$

From the expression for the second moment of the process, we obtain

$$\frac{\partial m_2}{\partial t} = \mathcal{L}^*_{x_1} m_2(t, x_1, x_2) + \mathcal{L}^*_{x_2} m_2(t, x_1, x_2) - 2\gamma^0_{2,2}(t)[m_2(t, x_1, x_2) - m(t, x_1)\delta_{x_1}(x_2)] \tag{27}$$

where $\mathcal{L}^*_{x_i}$ is the adjoint operator of $\mathcal{L}_{x_i}$ and $\gamma^0_{2,2}(s) = \frac{a}{2g(s)} + \frac{1}{2}\int_0^\infty \frac{u^2}{[g(s)+u]^2} \nu(du)$.

Denoting by $\xi = x_2 - x_1$ the difference between the types x_1 and x_2, we define the **mean density of types differing by** ξ as

$$I(t, \xi) = \int_{-\infty}^{+\infty} m_2(t, x, x + \xi) dx \tag{28}$$

that satisfy the following equation

$$\frac{\partial I}{\partial t} = \mathcal{L}^*_\xi I(t, \xi) - 2\gamma^0_{2,2}(t)[I(t, \xi) - \delta_0(\xi)]. \tag{29}$$

We restrict our attention to the Ohta-Kimura ladder model [10] and put $\mathcal{L} \equiv \frac{d^2}{dx^2}$. Fleming & Viot [6] showed that, when there is no jump and there is homogeneity in time, for large times $I(t, \xi)$ decays exponentially with ξ increasing. The compleat solution of (29), however, exhibit a variety of patterns, depending on the initial conditions. The solution $I(t, \xi)$ may be written as

$$I(t, \xi) = \int_{-\infty}^{+\infty} I(t_0, \xi - x) \frac{e^{-x^2/4(t-t_0) - 2\gamma^0_{2,2}(t_0, t)}}{\sqrt{2(t - t_0)}} dx + \int_{t_0}^{t} \frac{e^{-\xi^2/4(t-s) - 2\gamma^0_{2,2}(s,t)}}{\sqrt{2(t-s)}} \frac{2\gamma^0_{2,2}(s)}{\sqrt{2\pi}} ds. \tag{30}$$

The analysis of this function is left to the reader.

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