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Characterization of the stability boundary of nonlinear autonomous dynamical systems in the presence of a saddle-node equilibrium point of type 0

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Abstract: *Under the assumption that all equilibrium points are hyperbolic, the stability boundary of nonlinear autonomous dynamical systems is characterized as the union of the stable manifolds of equilibrium points on the stability boundary. The existing theory of characterization of the stability boundary is extended in this paper to consider the existence of non-hyperbolic equilibrium points on the stability boundary. In particular, a complete characterization of the stability boundary is presented when the system possesses a saddle-node equilibrium point of type* 0 *on the stability boundary. It is shown that the stability boundary of an asymptoticaly stable equilibrium point consists of the stable manifolds of all hyperbolic equilibrium points on the stability boundary and the stable manifold of the saddle-node equilibrium point of type* 0*.*

Keywords : *Stability region, basin of attraction, Stability boundary, Saddle-node bifurcation*

1 Introduction

The problem of determining stability regions (basin of attraction) of nonlinear dynamical systems is of fundamental importance for many applications in engineering and sciences [1], [3], [6]. Optimal estimates of the stability region can be obtained exploring the characterization of the stability boundary (the boundary of the stability region) [3].

Comprehensive characterizations of the stability boundary of classes of nonlinear dynamical systems can be found, for example, in [2]. The existing characterizations of stability boundary are proved under the key assumption that all the equilibrium points on the stability boundary are hyperbolic. In this paper, however, we are interested in the study of stability boundary when local bifurcations occur on the stability boundary and the assumption of hyperbolicity of equilibrium points is violated at bifurcation points. The characterization of the stability boundary at bifurcation points is of fundamental importance to understand how stability region behaves under parameter variation.

In this paper, we study the stability boundary characterization in the presence of a saddle-node bifurcation of type 0. Necessary and sufficient conditions for a saddle-node equilibrium point of type 0 lying on the stability boundary are presented. A complete characterization of the stability boundary when the system possesses a saddle-node equilibrium point of type 0 on the stability boundary is also presented. It is shown that the stability boundary consists of the stable manifolds of all hyperbolic equilibrium points on the stability boundary union with the stable manifold of the saddle-node equilibrium point of type 0 on the stability boundary.

2 Preliminaries on Dynamical Systems

In this section, classical concepts of the theory of dynamical systems are reviewed. More details on the content explored in this section can be found in [8]. Consider the nonlinear autonomous dynamical system

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a vector field of class C^r with $r \geq 1$. The solution of (1) starting at x at time $t = 0$ is denoted by $\varphi(t, x)$.

A point $x^* \in \mathbb{R}^n$ is an equilibrium point of (1) if $f(x^*) = 0$. An equilibrium point x^* of (1) is said to be *hyperbolic* if none of the eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial x}(x^*)$ has real part equal to zero. Moreover, a hyperbolic equilibrium point x^* is of *type* k if the Jacobian matrix possesses k eigenvalues with positive real part and $n - k$ eigenvalues with negative real part. A set $S \in \mathbb{R}^n$ is said to be an *invariant set* of (1) if every trajectory of (1) starting in S remains in S for all t.

Given an equilibrium point x^* of (1), the space $\mathbb{R}^n$ can be decomposed as a direct sum of three subspaces denoted by $E^s = span\ \{e_1, ..., e_s\}$, the stable subspace, $E^u = span\ \{e_{s+1}, ..., e_{s+u}\}$, the unstable subspace and $E^c = span\ \{e_{s+u+1}, ..., e_{s+u+c}\}$, the center subspace, with $s + u + c = n$, which are invariant with respect to the linearized system. The generalized eigenvectors $\{e_1, ..., e_s\}$ of the jacobian matrix associated with the eigenvalues that have negative real part span the stable subspace E^s while the generalized eigenvectors $\{e_{s+1}, ..., e_{s+u}\}$ and $\{e_{s+u+1}, ..., e_{s+u+c}\}$ respectively associated with the eigenvalues that have positive and zero real part span the unstable and center subspaces

If x^* is an equilibrium point of (1), then there exist local manifolds $W^s_{loc}(x^*)$, $W^{cs}_{loc}(x^*)$, $W^c_{loc}(x^*)$, $W^u_{loc}(x^*)$ and $W^{cu}_{loc}(x^*)$ of class C^r, which are invariant with respect to (1) [5]. These manifolds are tangent to E^s, $E^c \oplus E^s$, E^c, E^u and $E^c \oplus E^u$ at x^*, respectively, and are respectively called stable, stable center, center, unstable and unstable center manifolds. The stable and unstable manifolds are unique, but the stable center, center and unstable center manifolds may not be.

3 Saddle-node bifurcation

Consider the nonlinear dynamical system

$$\dot{x} = f(x, \lambda) \tag{2}$$

depending on the parameter $\lambda \in \mathbb{R}$. Let $f : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be a vector field of class C^r, with $r \geq 2$. For each fixed λ, one defines the vector field $f_\lambda = f(\cdot, \lambda)$ and $\varphi_\lambda(t, x)$ denotes the solution of $\dot{x} = f_\lambda(x)$ passing through x at time $t = 0$.

Definition 3.1. *A non-hyperbolic equilibrium point $x_{\lambda_0} \in \mathbb{R}^n$ of (2), for a fixed parameter $\lambda = \lambda_0$, is called a saddle-node equilibrium point and $(x_{\lambda_0}, \lambda_0)$ a saddle-node bifurcation point if the following conditions are satisfied:*
(SN1) $D_x f_{\lambda_0}(x_{\lambda_0})$ has a unique simple eigenvalue equal to 0 with v as an eigenvector to the right and w to the left.
(SN2) $w((\partial f_\lambda / \partial \lambda)(x_{\lambda_0}, \lambda_0)) \neq 0$.
(SN3) $w(D^2_x f_{\lambda_0}(x_{\lambda_0})(v, v)) \neq 0$.

Saddle-node equilibrium points or saddle-node bifurcation points can be classified in types according to the number of eigenvalues with positive real part.

Definition 3.2. *A saddle-node equilibrium point x_{λ_0} of (2), for a fixed parameter $\lambda = \lambda_0$, is called a saddle-node equilibrium point of type k and $(x_{\lambda_0}, \lambda_0)$ a saddle-node bifurcation point of type k if $D_x f_{\lambda_0}(x_{\lambda_0})$ has k eigenvalues with positive real part and $n-k-1$ with negative real part.*

The parameter value λ_0 of Definition 3.2 is called a saddle-node bifurcation value of type k.

In this paper, we will be mainly interested in saddle-node bifurcations of type 0. If x_{λ_0} is a saddle-node equilibrium point of type 0, then the following properties hold [9]:

(1) The unidimensional local center manifold $W^c_{loc,\lambda_0}(x_{\lambda_0})$ of x_{λ_0} can be splitted in three invariant submanifolds:

$$W^c_{loc,\lambda_0}(x_{\lambda_0}) = W^{c^-}_{loc,\lambda_0}(x_{\lambda_0}) \cup \{x_{\lambda_0}\} \cup W^{c^+}_{loc,\lambda_0}(x_{\lambda_0})$$

where $p \in W^{c^-}_{loc,\lambda_0}(x_{\lambda_0})$ implies $\varphi_{\lambda_0}(t,p) \longrightarrow x_{\lambda_0}$ as $t \longrightarrow +\infty$ and $p \in W^{c^+}_{loc,\lambda_0}(x_{\lambda_0})$ implies $\varphi_{\lambda_0}(t,p) \longrightarrow x_{\lambda_0}$ as $t \longrightarrow -\infty$.

(2) The (n-1)-dimensional local stable manifold $W^s_{loc,\lambda_0}(x_{\lambda_0})$ of x_{λ_0} exists, is unique and if $p \in W^s_{loc,\lambda_0}(x_{\lambda_0})$ then $\varphi_{\lambda_0}(t,p) \longrightarrow x_{\lambda_0}$ as $t \longrightarrow +\infty$.

(3) There is a neighborhood N of x_{λ_0} where the phase portrait of system (2) for $\lambda = \lambda_0$ on N is topologically equivalent to the phase portrait of Figure 1.

The center and stable manifolds are defined extending the local manifolds through the flow, i.e.:

$$W^s_{\lambda_0}(x_{\lambda_0}) := \bigcup_{t\leq 0} \varphi_{\lambda_0}(t, W^s_{loc,\lambda_0}(x_{\lambda_0})), \quad W^c_{\lambda_0}(x_{\lambda_0}) := W^{c^-}_{\lambda_0}(x_{\lambda_0}) \bigcup \{x_{\lambda_0}\} \bigcup W^{c^+}_{\lambda_0}(x_{\lambda_0})$$

where

$$W^{c^-}_{\lambda_0}(x_{\lambda_0}) := \bigcup_{t\leq 0} \varphi_{\lambda_0}(t, W^{c^-}_{loc,\lambda_0}(x_{\lambda_0})) \ \text{ and } \ W^{c^+}_{\lambda_0}(x_{\lambda_0}) := \bigcup_{t\geq 0} \varphi_{\lambda_0}(t, W^{c^+}_{loc,\lambda_0}(x_{\lambda_0})).$$

Obviously, $p \in W^s_{\lambda_0}(x_{\lambda_0})$ implies $\varphi_{\lambda_0}(t,p) \longrightarrow x_{\lambda_0}$ as $t \longrightarrow +\infty$, $p \in W^{c^-}_{\lambda_0}(x_{\lambda_0})$ implies $\varphi_{\lambda_0}(t,p) \longrightarrow x_{\lambda_0}$ as $t \longrightarrow +\infty$ and $p \in W^{c^+}_{\lambda_0}(x_{\lambda_0})$ implies $\varphi_{\lambda_0}(t,p) \longrightarrow x_{\lambda_0}$ as $t \longrightarrow -\infty$.

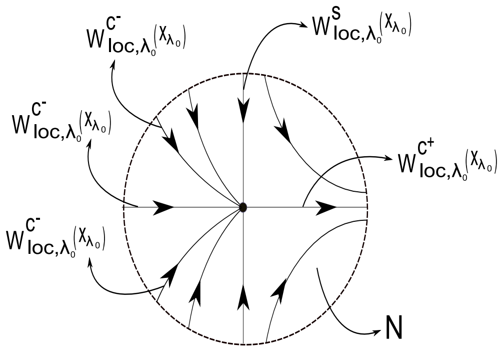

Figure 1: The phase portrait of system (2) in the neighborhood of a saddle-node equilibrium point of type 0. The local manifolds $W^{c^+}_{loc,\lambda_0}(x_{\lambda_0})$ and $W^s_{loc,\lambda_0}(x_{\lambda_0})$ are unique while there are infinite choices for $W^{c^-}_{loc,\lambda_0}(x_{\lambda_0})$. Three possible choices for $W^{c^-}_{loc,\lambda_0}(x_{\lambda_0})$ are indicated in this figure.

In order to obtain more insights about the dynamical behavior of system (2), for a fixed parameter $\lambda = \lambda_0$, in the neighborhood of a type 0 saddle-node equilibrium point x_{λ_o}, a neighborhood $U \subseteq N$ of x_{λ_o} will be decomposed into subsets $U^+_{\lambda_0}$ and $U^-_{\lambda_0}$. We define

$$U^-_{\lambda_0} := \{p \in U \ : \ \varphi_{\lambda_0}(t,p) \to x_{\lambda_0} \ as \ t \to \infty\} \ \text{ and } \ U^+_{\lambda_0} := U - U^-_{\lambda_0}.$$

For any neighborhood $U \subseteq N$ of x_{λ_0}, we obviously have $U = U^-_{\lambda_0} \cup U^+_{\lambda_0}$.

4 Stability Boundary Characterization

In this section, an overview of the existing body theory about characterization of the stability boundary of nonlinear dynamical systems is presented.

Suppose x^s is an asymptoticaly stable equilibrium point of (1). The *stability region* (or basin of attraction) of x^s is the set

$$A(x^s) = \{x \in \mathbb{R}^n \ : \ \varphi(t,x) \to x^s \ as \ t \to \infty\},$$

of all initial conditions $x \in \mathbb{R}^n$ whose trajectories converge to x^s when t tends to infinite.

The stability region $A(x^s)$ is an open and invariant set. Its closure $\overline{A(x^s)}$ is invariant and the *stability boundary* $\partial A(x^s)$ is a closed and invariant set. If $A(x^s)$ is not dense in $\mathbb{R}^n$, then $\partial A(x^s)$ is of dimension $n-1$.

The unstable equilibrium points that lie on the stability boundary $\partial A(x^s)$ play an essential role in the stability boundary characterization.

Let x^s be a hyperbolic asymptotically stable equilibrium point of (1) and consider the following assumptions:
(A1) All the equilibrium points on $\partial A(x^s)$ are hyperbolic.
(A2) The stable and unstable manifolds of equilibrium points on $\partial A(x^s)$ satisfy the transversality condition [4].
(A3) Every trajectory on $\partial A(x^s)$ approaches one of the equilibrium points as $t \to \infty$.

Assumptions (A1) and (A2) are generic properties of dynamical systems in the form of (1). In other words, they are satisfied for almost all dynamical systems in the form of (1) and, in practice, do not need to be verified. On the contrary, assumption (A3) is not a generic property of dynamical systems and needs to be checked. The existence of an energy function is a sufficient condition to satisfy assumption (A3) [2].

Next theorem provides necessary and sufficient conditions to guarantee that an equilibrium point lies on the stability boundary in terms of properties of its stable and unstable manifolds.

Theorem 4.1. *[2] Let x^s be a hyperbolic asymptotically stable equilibrium point of (1) and $A(x^s)$ be its stability region. If x^* is an equilibrium point of (1) and assumptions (A1)-(A3) are satisfied, then:*
(i) $x^* \in \partial A(x^s)$ if and only if $W^u(x^*) \cap A(x^s) \neq \emptyset$.
(ii) $x^* \in \partial A(x^s)$ if and only if $W^s(x^*) \subseteq \partial A(x^s)$.

Exploring Theorem 4.1, next theorem provides a complete characterization of the stability boundary $\partial A(x^s)$ in terms of the unstable equilibrium points lying on the stability boundary. It asserts that the stability boundary $\partial A(x^s)$ is the union of the stable manifolds of the equilibrium points on $\partial A(x^s)$.

Theorem 4.2. *[2] Let x^s be a hyperbolic asymptotically stable equilibrium point of (1) and $A(x^s)$ be its stability region. If assumptions (A1)-(A3) are satisfied, then:*

$$\partial A(x^s) = \bigcup_i W^s(x^i)$$

where x^i, $i = 1, 2, ...$ are the equilibrium points on $\partial A(x^s)$.

Theorem 4.2 provides a complete characterization of the stability boundary of system (1) under assumptions (A1)-(A3). In this work, we study the characterization of the stability boundary when local bifurcations occur at the stability boundary and assumption (A1) is violated. In particular, we study the stability boundary characterization when a saddle-node non-hyperbolic equilibrium point of type 0 lies on the stability boundary.

5 Saddle-node equilibrium point on the stability boundary

In this section, a complete characterization of the stability boundary in the presence of a saddle-node equilibrium point of type 0 is developed.

Next theorem offers necessary and sufficient conditions to guarantee that a saddle-node equilibrium point of type 0 lies on the stability boundary in terms of the properties of its stable and center manifolds. They also provide insight into how to develop a computational procedure to check if a saddle-node equilibrium point of type 0 lies on the stability boundary.

Theorem 5.1. ***(Saddle-node equilibrium point of type* 0 *on the stability boundary):*** *Let x_{λ_0} be a saddle-node equilibrium point of type 0 of (2), for $\lambda = \lambda_0$. Suppose also, for $\lambda = \lambda_0$, the existence of an asymptotically stable equilibrium point $x^s_{\lambda_0}$ and let $A_{\lambda_0}(x^s_{\lambda_0})$ be its stability region. Then the following holds:*
(i) $x_{\lambda_0} \in \partial A_{\lambda_0}(x^s_{\lambda_0})$ if and only if $W^{c^+}_{\lambda_0}(x_{\lambda_0}) \cap \overline{A_{\lambda_0}(x^s_{\lambda_0})} \neq \emptyset$.
(ii) $x_{\lambda_0} \in \partial A_{\lambda_0}(x^s_{\lambda_0})$ if and only if $(W^s_{\lambda_0}(x_{\lambda_0}) - \{x_{\lambda_0}\}) \cap \partial A_{\lambda_0}(x^s_{\lambda_0}) \neq \emptyset$.

Proof. (i) ($\Longleftarrow$) Suppose that $W^{c^+}_{\lambda_0}(x_{\lambda_0}) \cap \overline{A_{\lambda_0}(x^s_{\lambda_0})} \neq \emptyset$ and take $x \in W^{c^+}_{\lambda_0}(x_{\lambda_0}) \cap \overline{A_{\lambda_0}(x^s_{\lambda_0})}$. Note that $\varphi_{\lambda_0}(t,x) \longrightarrow x_{\lambda_0}$ as $t \longrightarrow -\infty$. On the other hand, set $\overline{A_{\lambda_0}(x^s_{\lambda_0})}$ is invariant, thus $\varphi(t,x) \in \overline{A_{\lambda_0}(x^s_{\lambda_0})}$ for all $t \leq 0$. As a consequence, $x_{\lambda_0} \in \overline{A_{\lambda_0}(x^s_{\lambda_0})}$. Since $x_{\lambda_0} \notin A_{\lambda_0}(x^s_{\lambda_0})$, we have that $x_{\lambda_0} \in \{\mathbb{R}^n - A_{\lambda_0}(x^s_{\lambda_0})\}$. Therefore, $x_{\lambda_0} \in \partial A_{\lambda_0}(x^s_{\lambda_0})$.
(i) ($\Longrightarrow$) Suppose that $x_{\lambda_0} \in \partial A_{\lambda_0}(x^s_{\lambda_0})$. Let $B(q,\epsilon)$ be a ball of radius ϵ centered at q for some $q \in W^{c^+}_{\lambda_0}(x_{\lambda_0})$ and $\epsilon > 0$. Consider a disk D of dimension $n-1$ contained in $B(q,\epsilon)$ and transversal to $W^{c^+}_{\lambda_0}(x_{\lambda_0})$ at q. As a consequence of λ-lema for non-hyperbolic equilibrium points [7], we can affirm that $\cup_{t\leq 0}\varphi_{\lambda_0}(t, B(q,\epsilon)) \supset U^+_{\lambda_0}$ where U is a neighborhood of x_{λ_0}. Since $x_{\lambda_0} \in \partial A_{\lambda_0}(x^s_{\lambda_0})$, we have that $U \cap A_{\lambda_0}(x^s_{\lambda_0}) \neq \emptyset$. On the other hand, $U^-_{\lambda_0} \cap A_{\lambda_0}(x^s_{\lambda_0}) = \emptyset$, thus $U^+_{\lambda_0} \cap A_{\lambda_0}(x^s_{\lambda_0}) \neq \emptyset$. Thus, there exists a point $p \in B(q,\epsilon)$ and a time t^* such that $\varphi_{\lambda_0}(t^*,p) \in A_{\lambda_0}(x^s_{\lambda_0})$. Since $A_{\lambda_0}(x^s_{\lambda_0})$ is invariant, we have that $p \in A_{\lambda_0}(x^s_{\lambda_0})$. Since ϵ can be chosen arbitrarily small, we can find a sequence of points $\{p_i\}$ with $p_i \in A_{\lambda_0}(x^s_{\lambda_0})$ for all $i = 1,2,...$ such that $p_i \longrightarrow q$ as $i \longrightarrow \infty$, that is, $q \in \overline{A_{\lambda_0}(x^s_{\lambda_0})}$. Since $q \in W^{c^+}_{\lambda_0}(x_{\lambda_0})$, we have that $W^{c^+}_{\lambda_0}(x_{\lambda_0}) \cap \overline{A_{\lambda_0}(x^s_{\lambda_0})} \neq \emptyset$.
The proof of (ii) is similar to the proof of (i) and will be omitted. □

With some additional assumptions a sharper result regarding saddle-node equilibrium points of type 0 on the stable boundary is obtained.
Let $x^s_{\lambda_0}$ be an asymptotically stable equilibrium point, x_{λ_0} be a saddle-node equilibrium point of type 0 of (2) for a fixed parameter $\lambda = \lambda_0$ and consider the following assumptions:
$(S1)$ All the equilibrium points on $\partial A_{\lambda_0}(x^s_{\lambda_0})$ are hyperbolic, except possibly for x_{λ_0}.
$(S2)$ The stable and unstable manifolds of equilibrium points on $\partial A_{\lambda_0}(x^s_{\lambda_0})$ satisfy the transversality condition.
$(S3)$ Every trajectory on $\partial A_{\lambda_0}(x^s_{\lambda_0})$ approaches one of the equilibrium points as $t \to \infty$.
$(S4)$ The stable manifold of equilibrium points on $\partial A_{\lambda_0}(x^s_{\lambda_0})$ and the manifold $W^{c^+}_{\lambda_0}(x_{\lambda_0})$ satisfy the transversality condition.

Under assumptions $(S1)$, $(S3)$ and $(S4)$, next theorem offers necessary and sufficient conditions which are sharper than conditions of Theorem 5.1, to guarantee that a saddle-node equilibrium point of type 0 lies on the stability boundary of nonlinear autonomous dynamical systems.

Theorem 5.2. ***(Further characterization of the saddle-node equilibrium point of type* 0 *on the stability boundary):*** *Let x_{λ_0} be a saddle-node equilibrium point of type 0 of (2) for $\lambda = \lambda_0$. Suppose also, for $\lambda = \lambda_0$, the existence of an asymptotically stable equilibrium point $x^s_{\lambda_0}$ and let $A(x^s_{\lambda_0})$ be its stability region. If assumptions $(S1)$, $(S3)$ and $(S4)$ are satisfied, then*
(i) $x_{\lambda_0} \in \partial A_{\lambda_0}(x^s_{\lambda_0})$ if and only if $W^{c^+}_{\lambda_0}(x_{\lambda_0}) \cap A_{\lambda_0}(x^s_{\lambda_0}) \neq \emptyset$.
(ii) $x_{\lambda_0} \in \partial A_{\lambda_0}(x^s_{\lambda_0})$ if and only if $W^s_{\lambda_0}(x_{\lambda_0}) \subset \partial A_{\lambda_0}(x^s_{\lambda_0})$.

The proof of Theorem 5.2 is analogous to the proof of Theorem 5.1 and will be omitted.

Under assumptions $(S1)$-$(S4)$ and exploring the results of Theorem 4.1 and Theorem 5.2, we obtain the next theorem.

Theorem 5.3. ***(Hyperbolic equilibrium points on the stability boundary)****: Let x_{λ_0} be a saddle-node equilibrium point of type 0 of (2) for a fixed parameter $\lambda = \lambda_0$. Suppose also, for $\lambda = \lambda_0$, the existence of an asymptotically stable equilibrium point $x^s_{\lambda_0}$ and let $A_{\lambda_0}(x^s_{\lambda_0})$ be its stability region. If assumptions $(S1)$-$(S4)$ are satisfied, then*
(i) a hyperbolic equilibrium point $x^*_{\lambda_0} \in \partial A_{\lambda_0}(x_{\lambda_0})$ if and only if $W^u_{\lambda_0}(x^*_{\lambda_0}) \cap A_{\lambda_0}(x^s_{\lambda_0}) \neq \emptyset$.
(ii) a hyperbolic equilibrium point $x^*_{\lambda_0} \in \partial A_{\lambda_0}(x^s_{\lambda_0})$ if and only if $W^s_{\lambda_0}(x^*_{\lambda_0}) \subset \partial A_{\lambda_0}(x^s_{\lambda_0})$.

Theorem 5.3 is a more general result than Theorem 4.1, since assumption (A1) used in the proof of Theorem 4.1 is relaxed.

Exploring the results of Theorem 5.2 and Theorem 5.3, next theorem provides a complete characterization of the stability boundary when a saddle-node equilibrium point of type 0 lies on $\partial A_{\lambda_0}(x^s_{\lambda_0})$.

Theorem 5.4. ***(Stability Boundary Characterization)****: Let $x^s_{\lambda_0}$ be an asymptotically stable equilibrium point of (2), for $\lambda = \lambda_0$ and $A_{\lambda_0}(x^s_{\lambda_0})$ be its stability region. Suppose also, for $\lambda = \lambda_0$, the existence of a type 0 saddle-node equilibrium point x_{λ_0} on the stability boundary $\partial A_{\lambda_0}(x^s_{\lambda_0})$. If assumptions $(S1)$-$(S4)$ are satisfied, then*

$$\partial A_{\lambda_0}(x^s_{\lambda_0}) = \bigcup_i W^s_{\lambda_0}(x_{\lambda_0^i}) \bigcup W^s_{\lambda_0}(x_{\lambda_0})$$

where $x_{\lambda_0^i}$, $i = 1, 2, ...$ are the hyperbolic equilibrium points on $\partial A_{\lambda_0}(x^s_{\lambda_0})$.

Proof. If the hyperbolic equilibrium point $x_{\lambda_0^i} \in \partial A_{\lambda_0}(x^s_{\lambda_0})$, then, from Theorem 5.3, we have that $W^s_{\lambda_0}(x_{\lambda_0^i}) \subset \partial A_{\lambda_0}(x^s_{\lambda_0})$. Since $x_{\lambda_0} \in \partial A_{\lambda_0}(x^s_{\lambda_0})$, we have that $W^s_{\lambda_0}(x_{\lambda_0}) \subset \partial A_{\lambda_0}(x^s_{\lambda_0})$ from Theorem 5.2. Therefore, $\cup_i W^s_{\lambda_0}(x_{\lambda_0^i}) \cup W^s_{\lambda_0}(x_{\lambda_0}) \subset \partial A_{\lambda_0}(x^s_{\lambda_0})$. On the other hand, by assumption $(S3)$, if $p \in \partial A(x^s_{\lambda_0})$, then $\varphi_{\lambda_0}(t, p) \longrightarrow x_{\lambda_0^i}$ or $\varphi_{\lambda_0}(t, p) \longrightarrow x_{\lambda_0}$ as $t \longrightarrow \infty$. Since $W^{c^-}_{\lambda_0}(x_{\lambda_0}) \cap \partial A_{\lambda_0}(x^s_{\lambda_0}) = \emptyset$ we can affirm that $p \in W^s_{\lambda_0}(x_{\lambda_0^i})$ or $p \in W^s_{\lambda_0}(x_{\lambda_0})$. Therefore $\partial A_{\lambda_0}(x^s_{\lambda_0}) \subset \cup_i W^s_{\lambda_0}(x_{\lambda_0^i}) \cup W^s_{\lambda_0}(x_{\lambda_0})$ and the theorem is proven. □

6 Example

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= x^2 + y^2 - 1 \\ \dot{y} &= x^2 - y + \lambda \end{aligned} \tag{3}$$

with $(x, y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

System (3) possesses, for $\lambda_0 = -1$, three equilibrium points; they are $x_{\lambda_0} = (0, -1)$, a saddle-node equilibrium point of type 0, $x^s_{\lambda_0} = (-1, 0)$, an asymptoticaly stable equilibrium point and $x^*_{\lambda_0} = (1, 0)$, a hyperbolic equilibrium point of type 1. Both the saddle-node equilibrium point of type 0 and the type-one equilibrium point $x^*_{\lambda_0} = (1, 0)$ belong to the stability boundary of $x^s_{\lambda_0} = (-1, 0)$. The stability boundary $\partial A_{\lambda_0}(-1, 0)$ is formed, according to Theorem 5.4, as the union of the stable manifold of the type-one hyperbolic equilibrium point $(1, 0)$ and the stable manifold of the saddle-node equilibrium point of type zero $(0, -1)$. See Figure 2.

7 Conclusions

Necessary and sufficient conditions for a saddle-node equilibrium point of type 0 lying on the stability boundary were presented in this paper. These conditions provide insight into how to develop a computational procedure to check if a saddle-node equilibrium point of type 0 lies on the stability boundary. A characterization of the stability region when the system possesses a saddle-node equilibrium point of type 0 on the stability boundary was developed for a class of nonlinear autonomous dynamical systems. This characterization is an important step to study the behavior of the stability boundary under parameter variation.

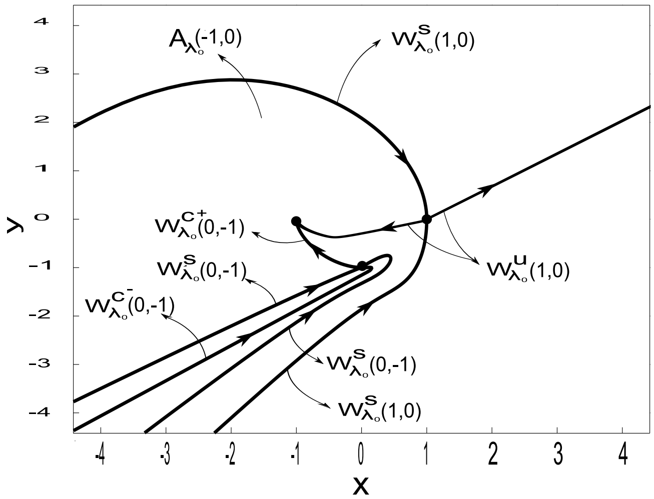

Figure 2: The phase portait of system (3) for $\lambda_0 = -1$.

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