

## Fixed point method and the existence of periodic solution for controlled nonlinear Ordinary Differential Equations

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***Abstract:** It is classical the use of topological degree techniques to perform qualitative analysis in evolution differential systems. Some of these methods are considered for the existence of periodic solutions in a non-linear Ordinary Differential Equation connected to the fixed points of an specific operator. However in general it is hard to deal with the operators involved in such connection. Here, we present the main tasks in this theory, and apply it for have the existence of periodic solutions in a Liénard class of controlled equations.*

### Introduction

Some of topological degree methods on fixed points are applied for consideration on the existence of periodic solutions in non-linear Ordinary Differential Equations. This fact can be seen for instance in [2], [3] and [4].

One of the techniques in this field is to make connection between the fixed point of a particular compact operator defined on a suitable space, and a periodic solution of the equation itself.

In the following we will be pointing this special operator in the theory and further applying it to a class of Liénard equations.

### Fixed points and periodic solutions

Let us consider the following non-linear T-periodic Differential Equation:

$$\dot{x} = A(t)x + f(t, x) \tag{1}$$

with

$$A: \mathbb{R} \rightarrow \mathbb{R}^{2n},$$

and

$$f: \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n,$$

where  $\Omega$  is an open subset of  $\mathbb{R}^n$  and  $A, f$  are continuous functions and T-periodic in the variable t.

Next we will be giving some definitions that are necessary for use in the following.

Let be  $C_T$  the set of all continuous T-periodic mappings

$$x : \mathbb{R} \rightarrow \mathbb{R}^n,$$

endowed with the sup norm  $\|x\|$ . In this way  $C_T$  is a Banach space.

Consider (2) the homogeneous equation associated to (1) and by (3) its adjoint equation,

$$\dot{x} = A(t)x \tag{2}$$

and

$$\dot{y} = A^t(t)y. \tag{3}$$

Define as usual the continuous operators – projections in fact – P and Q on  $C_T$ ,

$$P : C_T \rightarrow C_T \quad x \mapsto \sum_{1 \leq i \leq s} \langle x, \varphi_i \rangle \varphi_i$$

and

$$Q : C_T \rightarrow C_T \quad x \mapsto \sum_{1 \leq i \leq s} \langle x, \psi_i \rangle \psi_i,$$

where:

-  $\varphi_i$  and  $\psi_i$  ( $i=1,2,\dots,s$ ) are the elements of the basis for the periodic solutions of (2) and (3)

respectively,

and

- the functional  $\langle \cdot, \cdot \rangle$  on  $C_T$  is:

$$\langle x, y \rangle = \frac{1}{T} \int_0^T x^t(t)y(t)dt.$$

Now we are able to consider the compact operator  $M_f$  acting on  $C_T$  (for which the T-periodic solutions of (1) are exactly its fixed points). Namely:

$$M_f(x) = (P + QF + K(I - Q)F)(x),$$

where:

- I is the identity operator on  $C_T$ ,

- K:  $\text{Ker}(Q) \rightarrow \text{Ker}(P)$ , is the linear mapping that bounds to an element in  $\text{Ker}Q$  the unique T-periodic solution of (1) in  $\text{Ker}P$ . The existence of such unique solution of (1) is assured by the Fredholm alternative connecting properties of solutions of (1) and solutions of the homogeneous adjoint equation associated to (1),

and finally,

- the operator  $F$  on  $C_T$ ,  $Fx(t) = f(t, x(t))$ , is the Niemitskii's operator associated to  $f$ . As seen in [4] and [5] the set of all fixed points of  $M_f$  coincides with the  $T$ -periodic solutions of the equation (1).

In the next section we will be applying these results for a class of controlled Liénard equations.

### Existence of periodic solutions for Liénard equations

In the literature we have a huge amount of works considering the Liénard equation

$$\ddot{x} + h(x)\dot{x} + g(x) = u(t).$$

The most part concerns the autonomous case ( $u=0$ ) or the case in which  $u$  is an  $T$ -periodic non constant continuous function.

However, this kind of equations are very important for consideration on control theory in which case, the mappings  $u$  are not in general periodic nor continuous. For practical purposes, the function  $u$  it is in fact considered as of the relay type, assuming only two values  $k$  or  $-k$ .

When problems with minimal time strategy in the system are considered, the existence of cycles in both the cases

$$|u(t)| = k,$$

it is a fundamental task [1].

Here we consider this question and use the properties of the operator  $M$  defined in the precedent section.

We will be given in this section a result concerning the existence of  $T$ -periodic solutions for the Liénard equation

$$\ddot{x} + h(x)\dot{x} + g(x) = k \tag{4}$$

where  $h, g$  are continuous real  $T$ -periodic mappings and  $k$  is a positive constant real number.

The equation (4) it is:

$$\dot{x} = y - H(x) \qquad \dot{y} = -g(x) + k, \tag{5}$$

with  $H$  being

$$H(x) = \int_0^x h(s) ds.$$

Making  $u = \begin{pmatrix} x \\ y \end{pmatrix}$  the system (5) has the form of (1):

$$\dot{u} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u + \begin{pmatrix} -H(x) \\ -g(x) + k \end{pmatrix}, \tag{6}$$

with

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f(t, x) = \begin{pmatrix} -H(x) \\ -g(x) + k \end{pmatrix}.$$

The fundamental matrix of solutions for the homogeneous part of (6), is:

$$X(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \tag{7}$$

By other hand, the fundamental matrix of solutions for the adjoint equation becomes:

$$Y(t) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} . \quad (8)$$

Observe that the unique fundamental T-periodic solution for (7) and (8) are respectively

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Let us state now the main theorem in this work:

**Theorem1:** Consider the Liénard equation (5) and suppose that there exists an  $u_0 = (x_0, y_0)$  and positive real numbers  $a, b, c$  in such a way the following 4 inequalities holds:

- $|x_0| \leq a |u_0|$
- $\left\| \int_0^{x_0} g(s) ds \right\| \leq b |u_0|$
- $|k - g(x_0)| \leq c |u_0|$

and

- $a + (2T+1)b + (T^2+T)c < 1.$

Then, the equation (4) has a T-periodic solution.

Moreover, if

$$\int_0^T g(s) ds = k$$

such solution it is unique.

In fact: in the present case we have,

$$\begin{aligned} Mu(t) &= M \begin{pmatrix} x \\ y \end{pmatrix} (t) = \\ &= \begin{pmatrix} \frac{1}{T} \int_0^T x(t) dt - \int_0^t H(x(s)) ds - \frac{1}{T} \int_0^t \int_0^s H(x(r)) dr ds + \int_0^t \int_0^s (-g(x(r)) + k) dr ds \\ -\frac{1}{T} \int_0^T H(x(s)) ds + \int_0^t (-g(x(s)) + k) ds \end{pmatrix} \end{aligned}$$

Then we get:

$$\max_{0 \leq t \leq T} |Mu(t)| \leq \|x\| + (2T + 1)\|H(x)\| + (T^2 + T) \max_{0 \leq t \leq T} | -g(x(t)) + k |$$

Under the hypothesis in the theorem we have the last equality written as:

$$\|Mu_0\| \leq (a + (2T+1)b + (T^2+T)c) \|u_0\| < \|u_0\|.$$

According to the Banach fixed-point theorem, then M has at least a fixed point, that is: a T-periodic solution for (4).

To conclude the theorem we see that

$$\frac{1}{T} \int_0^T (-g(x(s)) + k) ds = \frac{1}{T} \int_0^T (-H(s), -g(x(s)) + k) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} ds.$$

Under the hypothesis that  $\int_0^T g(s) ds = k$ ,

we get that  $Fx \in \text{Ker} Q$ , for the Niemitskii's operator F. This shows us that the T-periodic solution in the system it is unique.

## Conclusions

Elsewhere [6] the authors compare these results with another ones about the existence of T-periodic solutions for Liénard equations in the literature. Furthermore, in such work it is, also, showed how to avoid the hypothesis on periodicity for f in (1).

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