Analyzing the relations between interval-valued D-Implications and interval-valued QL-Implications

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Abstract: *The aim of this work is to analyze the relationship between interval QL-implications and interval D-implications, studying some properties that relate these concepts. We also analyze under which conditions the main properties relating punctual D-implications and QL-implications are still valid when an interval-based fuzzy approach on best interval representations is considered.*

Palavras-chave: *Interval Fuzzy Logic, Interval Fuzzy Implication, D-implication, QL-implication*

1 Introduction

The interval-valued fuzzy set theory was introduced in an independent way by [14, 16, 26, 34] and considers the integration of Fuzzy Theory [33] and Interval Mathematics [21]. It has been studied from several viewpoints (see, e.g., [6, 10, 13, 17, 22, 31, 12]) in order to deal with the representation of different kinds of uncertainty in fuzzy logic. In this paper, an interval extension considers the best interval representation of a fuzzy membership function [6, 7] in order to model the uncertainty in the process of determining exact membership grades.

Among different models of fuzzy implications, *quantum logic implications* (QL-implications, for short) and *Dishkant implications* (D-implications, for short) were studied only recently (see, e.g., [18, 19]). Following the approach used in our previous works [3, 4, 5, 6, 7], the aim of this paper is to analyze the relationship between interval QL-implications [23] and interval D-implications [24], studying several important properties that relate these concepts. In particular, we analyze under which conditions the main properties relating punctual D-implications and QL-implications [18, 20] are still valid when an interval-based fuzzy approach is considered.

This paper is organized as follows. Section 2 reviews the main concepts related to interval representations. Interval fuzzy t-conorms (t-norms) and negations are presented in sections 3 and 4, respectively. Fuzzy implications, interval QL-implications and D-implications are discussed in the sequence. The interrelations between interval QL-implications and interval D-Implications are considered in Section 8. Section 9 is the Conclusion with some final remarks.

2 Interval Representations

Let $\mathbb{U} = \{[a, b] \mid 0 \leq a \leq b \leq 1\}$ be the set of subintervals of $U = [0, 1] \subseteq \mathbb{R}$. The *left* and *right projections* $l, r : \mathbb{U} \to U$ are defined by $l([a, b]) = a$ and $r([a, b]) = b$, respectively. For a given interval $X \in \mathbb{U}$, $l(X)$ and $r(X)$ are also denoted by $\underline{X}$ and $\overline{X}$, respectively.

Among the partial orders that may be defined on $\mathbb{U}$ [9], in this work we consider:

- *Product order*: $X \leq Y \Leftrightarrow \underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$ (the component-wise Kulisch-Miranker order).
- *Inclusion order*: $X \subseteq Y \Leftrightarrow \underline{X} \geq \underline{Y}$ and $\overline{X} \leq \overline{Y}$.

Definition 2.1 *$F : \mathbb{U}^n \longrightarrow \mathbb{U}$ is an interval representation of a function $f : U^n \longrightarrow U$ if, for each $\vec{X} \in \mathbb{U}^n$ and $\vec{x} \in \vec{X}$, $f(\vec{x}) \in F(\vec{X})$.* [27] [1].

$F : \mathbb{U}^n \longrightarrow \mathbb{U}$ is a better interval representation of $f : U^n \longrightarrow U$ than $G : \mathbb{U}^n \longrightarrow \mathbb{U}$, denoted by $G \sqsubseteq F$, if, for each $\vec{X} \in \mathbb{U}^n$, $F(\vec{X}) \subseteq G(\vec{X})$.

Definition 2.2 [27] *The best interval representation of a real function $f : U^n \longrightarrow U$ is the interval function $\widehat{f} : \mathbb{U}^n \longrightarrow \mathbb{U}$, defined by*

$$\widehat{f}(\vec{X}) = [\inf\{f(\vec{x}) \mid \vec{x} \in \vec{X}\}, \sup\{f(\vec{x}) \mid \vec{x} \in \vec{X}\}]. \quad (1)$$

The interval function $\widehat{f}$ is well defined and for any other interval representation F of f, $F \sqsubseteq \widehat{f}$. Thus, $\widehat{f}$ returns a narrower interval than any other interval representation of f and $\widehat{f}$ has the *optimality property* of interval algorithms [15], when it is seen as an algorithm to compute a real function f.

Remark 1 *The best interval representation of a function $f : U^n \longrightarrow U$ is the hull of the range of f, that is, for each $X \in \mathbb{U}^n$, $\widehat{f}(\vec{X}) = \{f(\vec{x}) : \vec{x} \in \vec{X}\} = f(\vec{X})$ if and only if f is continuous in the usual sense [27].*

Proposition 2.3 *Let $f : U^n \rightarrow U$ and $X_1, \ldots, X_n, Y_1, \ldots, Y_n \in \mathbb{U}$. If $X_i \subseteq Y_i$, for $i = 1 \ldots n$, then it holds that $\widehat{f}(X_1, \ldots, X_n) \subseteq \widehat{f}(Y_1, \ldots, Y_n)$.*
Proof: It is straightforward. ▲

3 Interval t-norms and t-conorms

A triangular conorm (norm), *t-conorm* (*t-norm*) for short, is a function $S(T) : U^2 \rightarrow U$ that is commutative, associative, monotonic and has 0 (1) as neutral element.

Example 3.1 *The maximum t-conorm and the minimum t-norm are expressed, respectively, as:*

$$S_M(x, y) = \max\{x, y\}, \quad T_M(x, y) = \min\{x, y\}.$$

In the following, we present the interval generalizations of t-conorms (t-norms), according to [6, 7].

Definition 3.2 *A function $\mathbb{S} : \mathbb{U}^2 \rightarrow \mathbb{U}$ is an interval t-conorm (t-norm) if it is commutative, associative, monotonic w.r.t. the product and inclusion order and $[0, 0]$ ($[1, 1]$) is the neutral element.*

Proposition 3.3 *[7, Theorem 4.3] If S (T) is a t-conorm (t-norm) then $\widehat{S}(\widehat{T}) : \mathbb{U}^2 \rightarrow \mathbb{U}$ is an interval t-conorm (t-norm).*

Characterizations of $\widehat{S}$ and $\widehat{T}$ are given, respectively, by:

$$\widehat{S}(X, Y) = [S(\underline{X}, \underline{Y}), S(\overline{X}, \overline{Y})], \quad (2)$$
$$\widehat{T}(X, Y) = [T(\underline{X}, \underline{Y}), T(\overline{X}, \overline{Y})]. \quad (3)$$

4 Interval Fuzzy Negations

A function $N : U \rightarrow U$ is a *fuzzy negation* if
N1 : $N(0) = 1$ and $N(1) = 0$;
N2 : If $x \geq y$ then $N(x) \leq N(y), \forall x, y \in U$.
Fuzzy negations satisfying the involutive property are called *strong fuzzy negations* [2]:
N3 : $N(N(x)) = x, \forall x \in U$.

Definition 4.1 *An interval function $\mathbb{N} : \mathbb{U} \longrightarrow \mathbb{U}$ is an interval fuzzy negation if, for any X, Y in $\mathbb{U}$, the following properties hold:*

[1] Observe that the concept of interval representation is different from interval extension and natural extension [21].

$\mathbb{N}1$: $\mathbb{N}([0,0]) = [1,1]$ *and* $\mathbb{N}([1,1]) = [0,0]$*;*
$\mathbb{N}2$: *If* $X \geq Y$ *then* $\mathbb{N}(X) \leq \mathbb{N}(Y)$*;*
$\mathbb{N}3$: *If* $X \subseteq Y$ *then* $\mathbb{N}(X) \subseteq \mathbb{N}(Y)$*.*

If $\mathbb{N}$ *also meets the involutive property, it is said to be a strong interval fuzzy negation:*

$\mathbb{N}4$: $\mathbb{N}(\mathbb{N}(X)) = X$, $\forall X \in \mathbb{U}$.

Let $N : U \longrightarrow U$ be a fuzzy negation. A characterization of $\widehat{N}$ is given by:

$$\widehat{N}(X) = [N(\overline{X}), N(\underline{X})]. \tag{4}$$

Proposition 4.2 $\mathbb{N} : \mathbb{U} \longrightarrow \mathbb{U}$ *is a strong interval fuzzy negation if and only if there exists a strong fuzzy negation* N *such that* $\mathbb{N} = \widehat{N}$ *[2].*

5 Fuzzy Implications

Several definitions for *fuzzy implications* have been given (see, e.g., [1, 8, 11, 25, 32]). The minimal properties that a fuzzy implication $I : U^2 \longrightarrow U$ must satisfy are the corner conditions:

$$I(1,1) = I(0,1) = I(0,0) = 1 \text{ and } I(1,0) = 0.$$

There are different classes of fuzzy implications obtained in a canonical way from other fuzzy connectives. This paper focuses on the classes of QL-implications and D-implications.

Let S be a t-conorm, N be a strong fuzzy negation and T be a t-norm. A QL-implication is a fuzzy implication defined, for all $x, y \in [0,1]$, by [28]:

$$I_{S,N,T}(x,y) = S(N(x), T(x,y)). \tag{5}$$

an a D-implication is a fuzzy implication defined, for all $x, y \in [0,1]$, by [18, 19]:

$$I_{S,T,N}(x,y) = S(T(N(x), N(y)), y). \tag{6}$$

The definitions of QL-implication and D-implication are closely related. In fact, a D-implication is just the contraposition with respect to N of a QL-implication, and the converse also holds. Thus, whenever I_D is a D-implication, I_{QL} is a QL-implication, and N is their underlying fuzzy negation, then we have that

$$D(I_{QL})(x,y) = I_{QL}(N(y), N(x)) \quad \text{and} \quad QL(I_D)(x,y) = I_D(N(y), N(x)) \tag{7}$$

are a D-implication and a QL-implication, respectively [19].

Proposition 5.1 *[18, Proposition 3][29, 30] Let* T *be a t-norm,* S *a t-conorm and* N *a strong fuzzy negation. If* $I_{S,N,T}$ *(or* $I_{S,T,N}$*) satisfies the property* $x \leq z \Rightarrow I(x,y) \geq I(z,y)$*, then*

$$S(x, N(x)) = 1, \textit{ for all } x \in U. \tag{8}$$

6 Interval QL-implications

An interval function $\mathbb{I} : \mathbb{U}^2 \rightarrow \mathbb{U}$ is an **interval QL-implication** if there are an interval t-conorm $\mathbb{S}$, a strong interval fuzzy negation $\mathbb{N}$ and an interval t-norm $\mathbb{T}$ such that

$$\mathbb{I}(X,Y) = \mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}(X,Y) = \mathbb{S}(\mathbb{N}(X), \mathbb{T}(X,Y)). \tag{9}$$

Proposition 6.1 *[23, Theorem 4] Let* S *be a t-conorm,* N *be a fuzzy negation and* T *be a t-norm. If* S*,* T *and* N *are continuous then*

$$\mathbb{I}_{\widehat{S},\widehat{N},\widehat{T}} = \widehat{I_{S,N,T}}. \tag{10}$$

Corollary 6.2 *[23, Corollary 1] If* I *is a continuous QL-Implication then* $\widehat{I}$ *is an interval QL-implication.*

Denote by $\mathcal{C}(S)$, $\mathcal{C}(T)$, $\mathcal{C}(I_{QL})$ and $\mathcal{C}(N)$ the classes of continuous t-conorms, t-norms and QL-implications, and strong fuzzy negations, respectively. The related interval extensions are indicated by $\mathcal{C}(\mathbb{S})$, $\mathcal{C}(\mathbb{T})$, $\mathcal{C}(\mathbb{I}_{\mathbb{QL}})$ and $\mathcal{C}(\mathbb{N})$, respectively. The results in previous sections and Proposition 6.1 state the commutativity of the diagram in Figure 1.

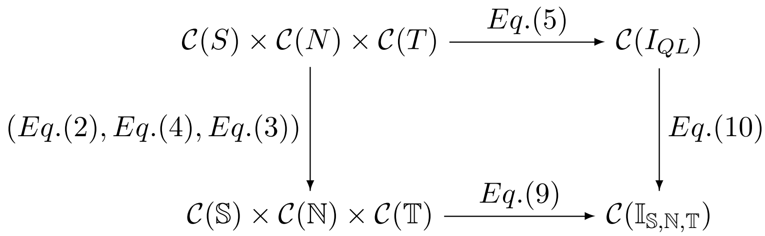

Figura 1: Commutative classes of interval QL-implications

7 Interval D-implications

An interval function $\mathbb{I} : \mathbb{U}^2 \to \mathbb{U}$ is an *interval D-implication* whenever there exist an interval t-conorm $\mathbb{S}$, an interval t-norm $\mathbb{T}$ and a strong interval fuzzy negation $\mathbb{N}$ such that

$$\mathbb{I}(X, Y) = \mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}(X, Y) = \mathbb{S}(\mathbb{T}(\mathbb{N}(X), \mathbb{N}(Y)), Y). \quad (11)$$

Proposition 7.1 *[24, Proposition 7.1] Let S be a t-conorm, T be a t-norm and N be a fuzzy negation. If S, T and N are continuous then*

$$\mathbb{I}_{\widehat{S},\widehat{T},\widehat{N}} = \widehat{I_{S,T,N}}. \quad (12)$$

Corollary 7.2 *[24, Corollary 7.2] If I is a continuous D-Implication then $\widehat{I}$ is an interval D-implication.*

Denote by $\mathcal{C}(S)$, $\mathcal{C}(T)$, $\mathcal{C}(I_D)$ and $\mathcal{C}(N)$ the classes of continuous t-conorms, t-norms and D-implications, and strong fuzzy negations, respectively. The related interval extensions are indicated by $\mathcal{C}(\mathbb{S})$, $\mathcal{C}(\mathbb{T})$, $\mathcal{C}(\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}})$ and $\mathcal{C}(\mathbb{N})$, respectively. The results presented in previous sections and Proposition 7.1 state the commutativity of the diagram in Figure 2.

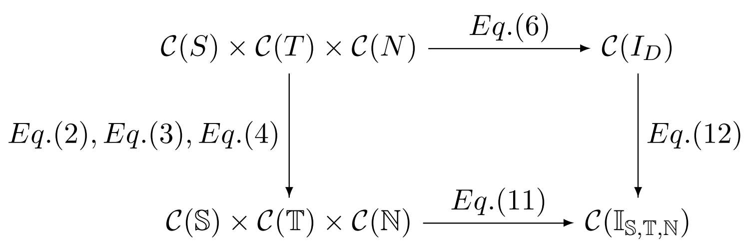

Figura 2: Commutative classes of interval D-implications

8 Relating Interval QL-implications and Interval D-Implications

The results presented in this section are based on the study on the interrelations between QL-implications and D-implications, which appeared only recently as subject of some important works (see, e.g., [18, 19]). In this paper, we analyze important properties connecting interval D-implications and interval QL-implications.

Proposition 8.1 *Let $\mathbb{T}$ be an interval t-norm, $\mathbb{S}$ an interval t-conorm and $\mathbb{N}$ a strong interval fuzzy negation. If $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ (or $\mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}$) satisfies the first place antitonicity property FPA: $X \leq Z \Rightarrow \mathbb{I}(X,Y) \geq \mathbb{I}(Z,Y)$ (correspondingly the second place isotonicity property SPI: $Y \leq Z \Rightarrow \mathbb{I}(X,Y) \leq \mathbb{I}(X,Z)$) then it holds that*

$$\mathbb{S}(X, \mathbb{N}(X)) = [1,1], \quad \textit{for all} \quad X \in \mathbb{U}. \quad (13)$$

Proof: Suppose that $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}$ satisfies the property FPA. Then, it follows that

$$\begin{aligned}
\mathbb{S}(X, \mathbb{N}(X)) &= \mathbb{S}(\mathbb{T}(X, [1,1]), \mathbb{N}(X)) \\
&= \mathbb{S}(\mathbb{N}(X), \mathbb{T}(X, [1,1])) \\
&= \mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}(X, [1,1]) \\
&\geq \mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}([1,1], [1,1]) \ \text{(by FPA)} \\
&= [1,1]
\end{aligned}$$

Thus, one has that $\mathbb{S}(X,\mathbb{N}(X))=[1,1]$. The other case is analogous, considering SPI instead of FPA. ▲

Proposition 8.2 *Let $\mathbb{I}_D$ be an interval D-implication, $\mathbb{I}_{QL}$ an interval QL-implication and $\mathbb{N}$ their underlying interval fuzzy negation. Then*

$$D(\mathbb{I}_{QL})(X,Y)=\mathbb{I}_{QL}(\mathbb{N}(Y),\mathbb{N}(X)) \text{ and } QL(\mathbb{I}_D)(X,Y)=\mathbb{I}_D(\mathbb{N}(Y),\mathbb{N}(X)) \quad (14)$$

are an interval D-implication and an interval QL-implication, respectively.

Proof: Suppose that $\mathbb{I}_{QL}$ is an interval QL-implication. Then, it follows that

$$\begin{aligned} D(\mathbb{I}_{QL})(X,Y) &= \mathbb{I}_{QL}(\mathbb{N}(Y),\mathbb{N}(X)) \\ &= \mathbb{S}(\mathbb{N}(\mathbb{N}(Y)),\mathbb{T}(\mathbb{N}(Y),\mathbb{N}(X))) \\ &= \mathbb{S}(Y,\mathbb{T}(\mathbb{N}(X),\mathbb{N}(Y))) \\ &= \mathbb{S}(\mathbb{T}(\mathbb{N}(X),\mathbb{N}(Y)),Y) \\ &= \mathbb{I}_{\mathbb{S},\mathbb{T},\mathbb{N}}(X,Y) \end{aligned}$$

Therefore, $D(\mathbb{I}_{QL})$ is an interval D-implication. The other case is analogous. ▲

Proposition 8.3 *Let T_M be the minimum t-norm presented in Example 3.1, $\mathbb{N}$ a strong interval fuzzy negation and $\mathbb{S}$ an interval t-conorm satisfying equation 13. Then, it holds that*

$$\mathbb{I}_{\mathbb{S},\mathbb{N},\widehat{T_M}}(X,Y)\subseteq\mathbb{I}_{\mathbb{S},\widehat{T_M},\mathbb{N}}(X,Y) = \begin{cases} [1,1] & \textit{if } X\leq Y, \\ \mathbb{S}(\mathbb{N}(X),Y) & \textit{if } X\geq Y; \\ \mathbb{S}(\mathbb{N}([\underline{X},\overline{Y}]),Y) & \textit{if } X\subseteq Y; \\ \mathbb{S}(\mathbb{N}([\underline{Y},\overline{X}]),X) & \textit{if } Y\subseteq X \end{cases} \quad (15)$$

and, for $I_{\mathbb{S},\widehat{T_M},\mathbb{N}}$, the first place antitonicity property holds.

Proof: Suppose that T_M is the minimum t-norm presented in Example 3.1, $\mathbb{N}$ is a strong interval fuzzy negation and $\mathbb{S}$ is an interval t-conorm satisfying equation 13. Then, it follows that
(i) If $X\leq Y$ then $\mathbb{I}_{\mathbb{S},\mathbb{N},\widehat{T_M}}(X,Y)=\mathbb{S}(\mathbb{N}(X),\widehat{T_M}(X,Y))=\mathbb{S}(X,\mathbb{N}(X))=[1,1]$ and $\mathbb{I}_{\mathbb{S},\widehat{T_M},\mathbb{N}}(X,Y)=\mathbb{S}(\widehat{T_M}(\mathbb{N}(X),\mathbb{N}(Y)),Y)=\mathbb{S}(\mathbb{N}(Y),Y)=[1,1]$.
(ii) If $Y\leq X$ then $\mathbb{I}_{\mathbb{S},\mathbb{N},\widehat{T_M}}(X,Y)=\mathbb{S}(\mathbb{N}(X),\widehat{T_M}(X,Y))=\mathbb{S}(\mathbb{N}(X),Y)$ and $\mathbb{I}_{\mathbb{S},\widehat{T_M},\mathbb{N}}(X,Y)=\mathbb{S}(\widehat{T_M}(\mathbb{N}(X),\mathbb{N}(Y)),Y)=\mathbb{S}(\mathbb{N}(X),Y)$.
(iii) If $X\subseteq Y$ then

$$\begin{aligned} \mathbb{I}_{\mathbb{S},\mathbb{N},\widehat{T_M}}(X,Y) &= \mathbb{S}(\mathbb{N}(X),\widehat{T_M}(X,Y)) \\ &= \mathbb{S}(\mathbb{N}(X),[\underline{Y},\overline{X}] \\ &\subseteq \mathbb{S}(\mathbb{N}(\underline{X},\overline{Y}),Y) \\ &= \mathbb{S}([N(\overline{Y}),N(\underline{X})],Y) \\ &= \mathbb{S}(\widehat{T_M}([N(\overline{X}),N(\underline{X})],[N(\overline{Y}),N(\underline{Y})]),Y) \\ &= \mathbb{S}(\widehat{T_M}(\widehat{N}(X),\widehat{N}(Y)),Y) \\ &= \mathbb{S}(\widehat{T_M}(\mathbb{N}(X),\mathbb{N}(Y)),Y) \\ &= I_{\mathbb{S},\widehat{T_M},\mathbb{N}}(X,Y). \end{aligned}$$

(iv) The proof for $\mathbb{S}(\mathbb{N}([\underline{Y},\overline{X}],X)$, if $Y\subseteq X$, is analogous to the case (iii).

The proof for the FPA property related to $I_{\mathbb{S},\widehat{T_M},\mathbb{N}}$ follows from the antitonicity of $\mathbb{N}$ and the monotonicity of $\widehat{T_M}$ and $\mathbb{S}$. ▲

Given any interval QL-implication or interval D-implication I, it holds that $\mathbb{I}(X,[0,0])=\mathbb{N}(X)$. Interrelations between interval QL-implications and interval D-Implications may be analyzed with respect to other properties. For example, if an interval QL-implication or interval D-implication $\mathbb{I}$ satisfies the exchange principle, that is, $\mathbb{I}(X,\mathbb{I}(Y,Z))=\mathbb{I}(Y,\mathbb{I}(X,Z))$, then it also satisfies the contrapositive symmetry with respect to a strong interval negation $\mathbb{N}$, that is:

$$\mathbb{I}(\mathbb{N}(X),\mathbb{N}(Y))=\mathbb{I}(\mathbb{N}(Y),\mathbb{I}(X,[0,0]))=\mathbb{I}(X,\mathbb{I}(\mathbb{N}(Y),[0,0]))=\mathbb{I}(X,Y).$$

Proposition 8.4 *Let $\mathbb{T}$ be an interval t-norm, $\mathbb{S}$ an interval t-conorm and $\mathbb{N}$ as an interval strong negation such that the corresponding interval QL-implication $\mathbb{I}_{QL}$ (equivalently, the interval D-implication $\mathbb{I}_D$) is an implication satisfying the property FPA (SPI). The following statements are equivalent:*

1. *$\mathbb{I}_Q$ satisfies the exchange principle;*
2. *$\mathbb{I}_D$ satisfies the exchange principle;*
3. *$\mathbb{I}_{QL}$ is an interval S-implication[2];*
4. *$\mathbb{I}_D$ is an interval S-implication;*
5. *There exists a t-conorm S_1 such that* $\mathbb{S}(\mathbb{N}(X), \mathbb{T}(X, Y)) = \mathbb{S}_1(\mathbb{N}(X), Y)$, *for all* $X, Y \in \mathbb{U}$.

Proof: See [18, Proposition 8]. ▲

9 Conclusion and Final Remarks

Among the usual fuzzy implications used in fuzzy rule based systems and for performing inferences in approximate reasoning and fuzzy control, this paper focused on the interval-valued QL-implications and their contrapositions with respect to strong interval fuzzy negations, namely the interval-valued D-implications constructed from two special aggregation operators: interval t-norms and interval t-conorms.

The paper mainly discussed under which conditions generalized fuzzy QL-implications and D-implications applied to interval values preserve properties of canonical forms. The significance for requiring that the interval QL-operators and D-operators to be also interval implications was emphasized, since then these operators can be used in fuzzy inference processes. It was shown that some properties of fuzzy logic (e.g, contrapositive symmetry, exchange principle, first place antitonicity, second place isotonicity) might be naturally extended for interval fuzzy degrees, considering the respective degenerate intervals. The study of the relationships between interval-valued D-Implications and interval-valued QL-Implications obtained from the action of an interval automorphism is ongoing work.

Acknowledgments. This work was partially supported by CNPq (Proc. 307879/06-2, 473201/07-0, 307185/07-9).

Referências

[1] J. Balasubramaniam, Yager's new class of implications J_f and some classical tautologies, *Information Sciences*, **177**(3) (2007) 930–946.

[2] B.C. Bedregal, On interval fuzzy negations, (2009) (submitted to *Fuzzy Sets and Systems*) .

[3] B.C. Bedregal, G.P. Dimuro, R.H.N. Santiago, R.H.S. Reiser, On interval fuzzy S-implications, (2009) (to appear in *Information Science*).

[4] B.C. Bedregal, R.H.N. Santiago, R.H.S. Reiser and G.P. Dimuro, Properties of fuzzy implications obtained via the interval constructor, *TEMA* **8**(1) (2007) 33–42.

[5] B.C. Bedregal, R.H.N. Santiago, R.H.S Reiser and G.P. Dimuro, The best interval representation of fuzzy S-implications and automorphisms, in "Proc. of IEEE Intl. Conf. on Fuzzy Systems" pp. 3220–3230, IEEE, Los Alamitos, 2007.

[6] B. Bedregal and A. Takahashi, The best interval representation of t-norms and automorphisms, *Fuzzy Sets and Systems* **157**(24) (2006) 3220–3230.

[7] B. Bedregal and A. Takahashi, Interval-valued versions of t-conorms, fuzzy negations and fuzzy implications, in "Proc. of IEEE Intl. Conf. on Fuzzy Systems" pp. 9553–9559, IEEE, Los Alamitos, 2006.

[8] H. Bustince, P. Burilo and F. Soria, Automorphism, negations and implication operators, *Fuzzy Sets and Systems* **134** (2003) 209–229.

[2] The notion of interval S-implication considered here is the same considered in [3].

[9] R. Callejas-Bedregal and B.C. Bedregal, Intervals as a domain constructor, *TEMA* **2** (2001) 43–52.

[10] D. Dubois and H. Prade, Interval-valued fuzzy sets, possibility theory and imprecise probability, in “Proc. of the Intl. Conf. on Fuzzy Logic and Technology” pp. 314–319, Barcelona, 2005.

[11] J.C. Fodor, On fuzzy implication operators, *Fuzzy Sets and Systems* **42** (1991) 293–300.

[12] B.V. Gasse, C. Cornelis, G. Deschrijver and E. Kerre, On the properties of a generalized class of t-norms in interval-valued fuzzy logics, *New Math. and Natural Computation* **2** (2006) 29–42.

[13] M. Gehrke, C. Walker and E. Walker, Some comments on interval valued fuzzy sets, *Intl. Journal of Intelligent Systems* **11** (1996) 751–759.

[14] I. Grattan-Guiness, Fuzzy membership mapped onto interval and many-valued quantities, *Z. Math. Logik. Grundladen Math.* 22 (1975) 149–160.

[15] T. Hickey, Q. Ju and M. Emdem, Interval arithmetic: from principles to implementation, *Journal of the ACM* **48**(5) (2001) 1038–1068.

[16] K. Jahn, Intervall-wertige mengen, *Math. Nach.* 68 (1975) 115–132.

[17] W.A. Lodwick, Preface, *Reliable Computing* **10**(4) (2004) 247–248.

[18] M. Mas, M. Monserrat and J. Torrens, QL-implications versus D-implications, *Kybernetika*, **42**(3) (2006) 351–366.

[19] M. Mas, M. Monserrat and J. Torrens, Two types of implications derived from uninorms, *Fuzzy Sets and Systems* **158**(3) (2007) 2612–2626.

[20] M. Mas, M. Monserrat and E. Trillas, A survey on fuzzy implication functions, *IEEE Transactions on Fuzzy Systems* **15**(6) (2007) 1107–1121.

[21] R. Moore, “Methods and Applications of Interval Analysis”, SIAM, Philadelphia, 1979.

[22] R. Moore and W. Lodwick, Interval analysis and fuzzy set theory, *Fuzzy Sets and Systems* **135**(1) (2003) 5–9.

[23] R.H.S. Reiser, G.P. Dimuro, B.C. Bedregal and R.H.N. Santiago, Interval-valued QL-implications, in “Logic, Language, Information” (D. Leivant and R. Queiroz, eds.), LNCS, Vol. 4576, pp. 307–321, Springer, Berlim, 2007.

[24] R.H.S. Reiser, G.P. Dimuro, B.R.C. Bedregal, Interval-valued D-implications, in “Anais XXXI CNMAC”, UNAMA/SBMAC, Belém, 2008. (submitted to TEMA)

[25] D. Ruan and E. Kerre, Fuzzy implication operators and generalized fuzzy methods of cases, *Fuzzy Sets and Systems* **54** (1993) 23–37.

[26] R. Sambuc, “Fonctions ϕ-floues. Application l’aide au Diagnostic en Pathologie Thyroidienne”, Ph.D. thesis, Univ. Marseille, Marseille, 1975.

[27] R.H.N. Santiago, B.C. Bedregal and B. Acióly, Formal aspects of correctness and optimality in interval computations, *Formal Aspects of Computing* **18**(2) (2006) 231–243.

[28] Y. Shi, B. Van Gasse, D. Ruan and E. E. Kerre, On the first place antitonicity in QL-implications, *Fuzzy Sets and Systems* **159** (2008) 2998–3013.

[29] E. Trillas, C. del Campo and S. Cubillo, When QM-operators are implication functions and conditionsl fuzzy relations, *Intl. Journal of Intelligent systems* **15**(2000) 647–655.

[30] E. Trillas, C. Alsina, E. Renedo and A. Pradera, On contra-symmetry and MPT conditionality in fuzzy logic, *Intl. Journal of Intelligent systems* **20**(2005) 313–326.

[31] I. Turksen, Fuzzy normal forms, *Fuzzy Sets and Systems* **69** (1995) 319–346.

[32] R. Yager, On some new classes of implication operators and their role in approximate reasoning, *Information Sciences* **167** (2004) 193–216.

[33] L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965) 338–353.

[34] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning - I, *Information Sciences* **6** (1975) 199–249.