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Hypergeometric functions and L-orthogonal polynomials

C.F. Bracciali, R.L. Lamblém e A. Sri Ranga
Depto de Ciências de Computação e Estatística, IBILCE
UNESP - Universidade Estadual Paulista
15054-000, São José do Rio Preto, SP
E-mail: parareginae@hotmail.com

Resumo: *We consider a ratio of hypergeometric functions that can be expressed in terms of a continued fraction expansion and obtain polynomials with L-orthogonal properties. An explicit expression for these polynomials in terms of hypergeometric polynomials is also found. Information on the parameters, when the L-orthogonality is of three special classes, are also observed.*

Palavras-chave: *Hypergeometric functions, continued fractions, L-orthogonal polynomials.*

1 Introduction

Hypergeometric series are given by

$$ {}_pF_q\left(\begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array}; z\right) = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j \cdots (a_p)_j}{(b_1)_j (b_2)_j \cdots (b_q)_j} \frac{z^j}{j!}, $$

where $(a)_n = a(a+1)\cdots(a+n-1)$.

Clearly, for any of the denominator parameters b_k a negative integer value must be avoided. Moreover, if any of the numerator parameters a_k is given the value $-n$ then series terminates at the n-th term, resulting in a polynomial of degree n.

Hypergeometric series with $p = 2$ and $q = 1$ are of special interest and are usually written as

$$ {}_2F_1\left(\begin{array}{c} a, b, \\ c \end{array}; z\right) \quad \text{or} \quad {}_2F_1(a, b; c; z). $$

This series is known to converge for $|z| < 1$. The analytic function represented by this series is usually known as the hypergeometric function. For example, when $\mathfrak{Re}(c) > \mathfrak{Re}(b) > 0$, this function has the integral representation (due to Euler)

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, $$

which holds for $z \notin [1, \infty)$. Here $\Gamma(z)$ is the gamma function.

Hypergeometric functions are important because almost all of the elementary mathematical functions are simply hypergeometric functions or ratios of hypergeometric functions.

Using the series expansion of ${}_2F_1(a, c-b; c; z)$, an analytic extension of the Hypergeometric function for $\mathfrak{Re}(z) < 1/2$ can be obtained by the Pfaff transformation

$$ {}_2F_1(a, b; c; z) = (1-z)^{-a} \, {}_2F_1(a, c-b; c; z/(z-1)). $$

Two hypergeometric functions are called contiguous if two of their corresponding parameters are pairwise identical and the other parameter differing by unity. There are some interesting relations between contiguous hypergeometric functions, called contiguous relations. We consider the following two contiguous relations (see (2.5.3) and (2.5.16) of [1]):

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \Big(1+\frac{a-b+1}{c}z\Big)\,{}_2F_1(a+1,b;c+1;z) \\ &\qquad\qquad -\frac{(a+1)(c-b+1)}{c(c+1)}z\,{}_2F_1(a+2,b;c+2;z) \end{aligned} \tag{1}$$

and

$$\begin{aligned} (c-a)\,{}_2F_1(a-1,b;c;z) &= \big(c-2a-(b-a)z\big)\ {}_2F_1(a,b;c;z) \\ &\qquad + a(1-z)\ {}_2F_1(a+1,b;c;z). \end{aligned} \tag{2}$$

In both cases one must assume $c \neq 0, -1, -2, \ldots$. We use these relations to obtain a class of L-orthogonal polynomials with interesting properties.

2 L-orthogonal polynomials

Given the sequences of non-zero complex numbers $\{\beta_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=2}^{\infty}$, let $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ be the sequence of polynomials given by the three term recurrence relations

$$\begin{aligned} P_{n+1}(z) &= (z+\beta_{n+1})P_n(z) - \alpha_{n+1}zP_{n-1}(z), \\ Q_{n+1}(z) &= (z+\beta_{n+1})Q_n(z) - \alpha_{n+1}zQ_{n-1}(z), \end{aligned} \qquad n \geq 1,$$

with $P_0(z) = 0$, $Q_0(z) = 1$, $P_1(z) = \mu_0 \neq 0$ and $Q_1(z) = z + \beta_1$.

Properties of these polynomials from the point of view of the recurrence coefficients are explored in many papers, including [4] and [6].

One can establish that there are formal power series expansions $L_0(z) = \sum_{j=0}^{\infty} -\mu_{-j-1}z^j$ and $L_\infty(z) = \sum_{j=1}^{\infty} \mu_{j-1}z^{-j}$, such that (see [2], [4])

$$\begin{aligned} L_0(z) - \frac{P_n(z)}{Q_n(z)} &= \frac{\mu_0\alpha_2\cdots\alpha_{n+1}}{\beta_1^2\cdots\beta_n^2\beta_{n+1}}z^n + O(z^{n+1}), \\ L_\infty(z) - \frac{P_n(z)}{Q_n(z)} &= \mu_0\alpha_2\cdots\alpha_{n+1}z^{-n-1} + O((1/z)^{n+2}), \end{aligned} \qquad n \geq 1. \tag{1}$$

If the determinants Δ_n and $\Delta_n^{(-1)}$ are defined by $\Delta_0 = \mu_0$, $\Delta_0^{(-1)} = \mu_{-1} = -\mu_0/\beta_1$,

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\ \vdots & \vdots & & \vdots \\ \mu_{-n} & \mu_{-n+1} & \cdots & \mu_0 \end{vmatrix} \quad \text{and} \quad \Delta_n^{(-1)} = \begin{vmatrix} \mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\ \mu_{-2} & \mu_{-1} & \cdots & \mu_{n-2} \\ \vdots & \vdots & & \vdots \\ \mu_{-n-1} & \mu_{-n} & \cdots & \mu_{-1} \end{vmatrix},$$

for $n \geq 1$, then applications of Cramer's rule to the linear system of equations that results from (1), give

$$\frac{\Delta_n}{\Delta_{n-1}} = \frac{\mu_0\alpha_2\cdots\alpha_{n+1}}{\beta_1\beta_2\cdots\beta_n} \quad \text{and} \quad \frac{\Delta_n^{(-1)}}{\Delta_n} = \frac{(-1)^{n+1}}{\beta_1\cdots\beta_n\beta_{n+1}}, \quad n \geq 1.$$

Hence, these determinants satisfy the conditions

$$\Delta_n \neq 0 \quad \text{and} \quad \Delta_n^{(-1)} \neq 0, \quad n \geq 0. \tag{2}$$

Note that, if $\alpha_{n+1}/\beta_n > 0$, $n \geq 1$, then $\Delta_n > 0$, $n \geq 0$ with the choice $\mu_0 > 0$.

Now from the series expansions L_0 and L_∞, that is from the double sequence $\{\mu_n\}_{n=-\infty}^{\infty}$, if we define a linear functional $\mathcal{M}$ by

$$\mathcal{M}[w^n] = \mu_n, \quad n = 0, \pm 1, \pm 2, \dots \ , \tag{3}$$

then

$$\mathcal{M}[w^{-n+s} Q_n(w)] = \mu_0 \alpha_2 \cdots \alpha_{n+1} \delta_{n,s}, \quad 0 \le s \le n-1, \quad n \ge 1. \tag{4}$$

We say the numbers μ_n, $n = 0, \pm 1, \pm 2, \dots$ are moments, $\mathcal{M}$ is the strong moment functional associated with these moments and that the sequence of polynomials $\{Q_n\}_{n=0}^{\infty}$ is a sequence of L-orthogonal polynomials with respect to this moment functional. The word L-orthogonality is used because (4) is equivalent to the orthogonality of the sequence of Laurent polynomials

$$Q_0(z),\ z^{-1}Q_1(z),\ z^{-1}Q_2(z),\ z^{-2}Q_3(z),\ z^{-2}Q_4(z),\ z^{-3}Q_5(z),\ \dots,$$

with respect to the same moment functional.

Given any double sequence $\{\mu_n\}_{n=-\infty}^{\infty}$, we will also refer to the strong moment functional defined by (3) as a quasi-definite strong moment functional if the conditions in (2) hold. With (2), the monic polynomials Q_n, $n \ge 0$, defined by

$$\mathcal{M}[w^{-n+s} Q_n(w)] = 0, \quad 0 \le s \le n-1,$$

exit and satisfy $Q_n(0) \ne 0$, $n \ge 1$ (see [4]).

There are three subclasses of strong quasi-definite moment functionals which are more interesting, useful and appear more frequently. We define them as

- Class 1A: $\mu_{-n} = \overline{\mu}_n, \quad \Delta_n \ne 0 \quad \text{and} \quad \Delta_n^{(-1)} \ne 0, \quad n \ge 0;$
- Class 1B: $\mu_{-n} = \mu_n, \quad \Delta_n \ne 0 \quad \text{and} \quad \Delta_n^{(-1)} \ne 0, \quad n \ge 0;$
- Class 2: $\mu_{-n-1} = -\mu_n \quad \text{and} \quad \Delta_n \ne 0, \quad n \ge 0.$

We refer to class 1A as the Szegő type class. The so called reciprocal polynomials $S_n(z) = Q_n^*(z) = z^n \overline{Q_n(1/\overline{z})}$, are the polynomials satisfying the Szegő orthogonality

$$\mathcal{M}[w^{-s} S_n(w)] = 0, \quad 0 \le s \le n-1.$$

Here, if one also assumes that $\Delta_n > 0$, $n \ge 0$, then the moment functional can be called a strong positive definite moment functional and it is well known that there exists a distribution function $\psi(e^{i\theta})$, defined in $[0, 2\pi]$, such that

$$\mu_n = \mathcal{M}[w^n] = \int_{\mathcal{C}} w^n d\psi(w) = \int_0^{2\pi} e^{in\theta} d\psi(e^{i\theta}), \quad n = 0, \pm 1, \pm 2, \dots, \quad .$$

In this case, for information about the polynomials S_n, simply known as Szegő polynomials, we cite the classical book [8] of Szegő and the recent book [7] of Simon.

In class 1B, which we may call the modified Szegő type class, the reversed polynomials $S_n(z) = Q_n^{\bullet}(z) = z^n Q_n(1/z)$ are the polynomials satisfying the Szegő orthogonality

$$\mathcal{M}[w^{-s} S_n(w)] = 0, \quad 0 \le s \le n-1.$$

When dealing real moments, hence with real polynomials, both classes 1A and 1B coincide.

In the definition of class 2, which we may call the para-orthogonal class, it is not necessary to write down the term $\Delta_n^{(-1)} \ne 0$ as it terns out $\Delta_n^{(-1)} = (-1)^{n+1} \Delta_n$ for $n \ge 0$. In this case, the polynomials $Q_n(z)$ are the para-orthogonal polynomials, $[S_n(z) + S_n^{\bullet}(z)]/[S_n(0) + S_n^{\bullet}(0)]$ or $[S_n(z) - S_n^{\bullet}(z)]/[S_n(0) - S_n^{\bullet}(0)]$, of some polynomials S_n satisfying the Szegő orthogonality with respect to another strong quasi-definite moment functional, say $\mathcal{N}$. The moment functional $\mathcal{N}$ is of class 1B.

We can also characterize all three classes in terms of the coefficients of the three term recurrence relation.

- Class 1A: $|\beta_1\beta_2\cdots\beta_n|^2 = \dfrac{\beta_n}{\beta_n - \alpha_{n+1}}, \quad n \geq 1;$

- Class 1B: $[\beta_1\beta_2\cdots\beta_n]^2 = \dfrac{\beta_n}{\beta_n - \alpha_{n+1}}, \quad n \geq 1;$

- Class 2: $\beta_n = 1, \quad n \geq 1.$

Moreover, the moment functional belonging to class 1A is a strong positive definite moment functional only if $\alpha_{n+1}/\beta_n > 0$, $n \geq 1$.

3 L-orthogonal polynomials from hypergeometric functions

Let $c - b \neq 0, -1, -2, \ldots$. Then from the contiguous relation (1),

$$\frac{{}_2F_1(a+1,-b+1;c-b+1;z)}{{}_2F_1(a,-b+1;c-b;z)} = \frac{1}{1+\dfrac{a+b}{c-b}z - \dfrac{(a+1)c}{(c-b)(c-b+1)}z\dfrac{{}_2F_1(a+2,-b+1;c-b+2;z)}{{}_2F_1(a+1,-b+1;c-b+1;z)}}.$$

Hence, if we write $R_n^{(a,b,c)}(z) = \dfrac{{}_2F_1(a+n+1,-b+1;c-b+n+1;z)}{{}_2F_1(a+n,-b+1;c-b+n;z)}$, $n = 0, 1, 2, \ldots$, then

$$R_0^{(a,b,c)}(z) = \frac{1}{1+g_1z} - \frac{f_2z}{1+g_2z} - \cdots - \frac{f_{n-1}z}{1+g_{n-1}z} - \frac{f_nz}{1+g_nz - f_{n+1}zR_n^{(a,b,c)}(z)},$$

where

$$g_n = \frac{a+b+n-1}{c-b+n-1}, \quad f_{n+1} = \frac{(a+n)(c+n-1)}{(c-b+n-1)(c-b+n)}, \quad n \geq 1.$$

If we restrict ourselves to the case in which $a = 0$, then

$$\begin{aligned} R_0^{(0,b,c)}(z) &= {}_2F_1(1,-b+1;c-b+1;z) \\ &= \frac{1}{1+g_1z} - \frac{f_2z}{1+g_2z} - \cdots - \frac{f_{n-1}z}{1+g_{n-1}z} - \frac{f_nz}{1+g_nz - f_{n+1}zR_n^{(0,b,c)}(z)}, \end{aligned}$$

where

$$g_n = \frac{b+n-1}{c-b+n-1}, \quad f_{n+1} = \frac{n(c+n-1)}{(c-b+n-1)(c-b+n)}, \quad n \geq 1.$$

Equivalently, we can also write

$$\begin{aligned} R_0^{(0,b,c)}(z)/\beta_1 &= \frac{1}{z+\beta_1} - \frac{\alpha_2z}{z+\beta_2} - \cdots - \frac{\alpha_nz}{z+\beta_n - \alpha_{n+1}z\,R_n^{(0,b,c)}(z)/\beta_{n+1}}, \\ &= \frac{z^{-1}}{1+\beta_1z^{-1}} - \frac{\alpha_2z^{-1}}{1+\beta_2z^{-1}} - \cdots - \frac{\alpha_nz^{-1}}{1+\beta_nz^{-1} - \alpha_{n+1}\,R_n^{(0,b,c)}(z)/\beta_{n+1}}, \end{aligned}$$

where

$$\beta_n = \frac{1}{g_n} = \frac{c-b+n-1}{b+n-1}, \quad \alpha_{n+1} = \frac{f_{n+1}}{g_ng_{n+1}} = \frac{n(c+n-1)}{(b+n-1)(b+n)}, \quad n \geq 1,$$

provided that $b \neq 0, -1, -2, \ldots$.

From the theory of continued fractions (see [3], [5]) the rational functions

$$\frac{P_n(z)}{Q_n(z)} = \frac{1}{\lfloor z+\beta_1} - \frac{\alpha_2 z}{\lfloor z+\beta_2} - \cdots - \frac{\alpha_n z}{\lfloor z+\beta_n}, \quad n \geq 1,$$

are such that

$$\begin{aligned} P_{n+1}(z) &= (z+\beta_{n+1})P_n(z) - \alpha_{n+1} z P_{n-1}(z), \\ Q_{n+1}(z) &= (z+\beta_{n+1})Q_n(z) - \alpha_{n+1} z Q_{n-1}(z), \end{aligned} \qquad n \geq 1,$$

with $P_0(z) = 0$, $Q_0(z) = 1$, $P_1(z) = 1$ and $Q_1(z) = z + \beta_1$. Hence, there exist $L_0(z) = \sum_{j=0}^{\infty} -\mu_{-j-1} z^j$ and $L_\infty(z) = \sum_{j=1}^{\infty} \mu_{j-1} z^{-j}$, with $\mu_0 = 1$, such that

$$\begin{aligned} L_0(z) - \frac{P_n(z)}{Q_n(z)} &= \frac{\mu_0 \alpha_2 \cdots \alpha_{n+1}}{\beta_1^2 \cdots \beta_n^2 \beta_{n+1}} z^n + O(z^{n+1}), \\ L_\infty(z) - \frac{P_n(z)}{Q_n(z)} &= \mu_0 \alpha_2 \cdots \alpha_{n+1} z^{-n-1} + O((1/z)^{n+2}), \end{aligned} \qquad n \geq 1.$$

Moreover, there exists a strong quasi-definite moment functional $\mathcal{M}$ such that $\mathcal{M}[w^n] = \mu_n$, $n = 0, \pm 1, \pm 2, \ldots$ and

$$\mathcal{M}[w^{-n+s} Q_n(w)] = \frac{(c)_n n!}{(b)_n (b+1)_n} \delta_{n,s}, \quad 0 \leq s \leq n-1, \quad n \geq 1.$$

Again from the theory of continued fractions we can identify that

$$L_0(z) = \sum_{j=0}^{\infty} -\mu_{-j-1} z^j = \beta_1^{-1} \, {}_2F_1(1, -b+1; c-b+1; z)$$

and thus $\mu_{-j} = (-b)_j / (c-b)_j$.

We now use the contiguous relation (2) to obtain an explicit expression for the above denominator polynomials Q_n. With $a = -n$ and z replaced by $1-z$, we have

$$\begin{aligned} {}_2F_1(-(n+1), b; c; 1-z) \;&= \Big(\frac{b+n}{c+n} z + \frac{c-b+n}{c+n}\Big) \, {}_2F_1(-n, b; c; 1-z) \\ &\qquad - \frac{n}{c+n} z \, {}_2F_1(-(n-1), b; c; 1-z), \quad n \geq 1. \end{aligned}$$

Since ${}_2F_1(0, b; c; 1-z) = 1$ and $\frac{c}{b} \, {}_2F_1(-(n-1), b; c; 1-z) = z + \frac{c-b}{b}$, by comparing the three term recurrence relations,

$$Q_n(z) = \frac{(c)_n}{(b)_n} \, {}_2F_1(-n, b; c; 1-z), \quad n \geq 0.$$

We easily note that the moment functional is of class 2 if $c - b + n - 1 = b + n - 1$, $n \geq 1$, which is equivalent to $c = 2b$.

Now, the moment functional is of class 1B if

$$\frac{[(c-b)_n]^2}{[(b)_n]^2} = \frac{(c-b+n-1)(b+n)}{(c-b-1)b}, \quad n \geq 1,$$

which leads to $c = 2b + 1$.

Finally, the moment functional is of class 1A if

$$\frac{|(c-b)_n|^2}{|(b)_n|^2} = \frac{(c-b+n-1)(b+n)}{(c-b-1)b}, \quad n \geq 1.$$

This is true if $c = 2\mathfrak{Re}(b) + 1$. It turns out in this case,

$$\frac{\alpha_{n+1}}{\beta_n} = \frac{n(n + 2\mathfrak{Re}(b))}{|b+n|^2} > 0, \quad n \geq 1,$$

and thus the associated moment functional is a strong positive definite moment functional.

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