

# NON-SPHERICITY OF THE MOON AND CRITICAL INCLINATION

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**Abstract.** *In this work, we consider the problem of a lunar artificial satellite perturbed by the non-uniform distribution of mass of the Moon taking into account the oblateness ( $J_2$ ) and the equatorial ellipticity (sectorial term  $C_{22}$ ). Using Lie-Hori method up to the second order it is eliminated the terms of short period of the Hamiltonian. A study is done for the critical inclination in first and second order of the disturbing potential analyzing the coupling terms of the non-uniform distribution of mass of the Moon. Numerical integrations are presented with the disturbing potential of first and second order.*

**Keywords:** Lunar artificial satellite, orbital perturbations, critical inclination.

## 1. INTRODUCTION

In this paper, we consider the problem of a lunar artificial satellite of low altitude taking into account the oblateness ( $J_2$ ) and the equatorial ellipticity (sectorial term  $C_{22}$ ) of the Moon. The Lie-Hori [15] perturbation theory method up to the second order is applied to eliminate the terms of short period of the disturbing potential. The perturbation method up to the second order is applied to analyze coupling terms that appear in the application of the method. In this work the term of long period of the disturbing potential is analyzed. A formula is developed for the calculation of the critical inclination when it is considered perturbations due to the non-sphericity of the Moon as a function of the terms of the zonal and sectorial harmonics.

In [9-10] and [11] is developed an analytical theory to study the orbital motion of lunar artificial satellites using the method of transformation of Lie ([8] and [13]) as a perturbation method. The main perturbation is due to the non-spherical gravitational field of the Moon and the attraction of the Earth. The disturbing body is in circular orbit with the expanded disturbing function in polynomial of Legendre up to the second order. In this paper is done an approach based on [11] where is considered only the non-sphericity of the Moon to study in first and second order the effect caused in satellites of low altitude. In [3-6] is developed an analytical theory with numerical applications taking in account the non-uniform distribution of mass of the Moon and the perturbation of the third body in elliptical orbit (Earth is considered). The disturbing function is expanded in Legendre polynomials up to the fourth order.

## 2. NON-SPHERICITY OF THE MOON

To analyze the motion of an artificial satellite of the Moon it is necessary to take into account the Moon's oblateness. Besides that the Moon is much less flattened than the Earth it also causes perturbations in space vehicles. Table 1 presents orders of magnitude for the zonal and sectorial harmonic to do a comparison with the same parameters for the Earth. The term  $C_{20}$  describes the equatorial bulge of the Moon, often referred to as the oblateness. The coefficient  $C_{22}$  measures the ellipticity of the equator.

Table 1- Magnitude orders for  $J_2$  and  $C_{22}$

	$C_{20} \equiv -J_2$	$C_{22}$
Earth	$-10^{-3}$	$2 \times 10^{-6}$
Moon	$-2 \times 10^{-4}$	$2 \times 10^{-5}$

The following assumptions have been done ([9] and [14]): a) the motion of the Moon is uniform (librations are neglected), b) the lunar equator lies in the ecliptic, c) we neglected the inclination of the lunar equator for the ecliptic (about  $1.5^\circ$ ), and the inclination of the lunar orbit to the ecliptic (about  $5^\circ$ ) and d) the longitude of the lunar longest meridian of the Moon is equal to the mean longitude of the Earth (librations are neglected). The order of the largest perturbation due to the Earth is about  $4 \times 10^{-5}$ , while that due to the Sun is approximately  $2 \times 10^{-7}$  ([19]) therefore, the perturbation of the Sun can be neglected.

We will go to present the Hamiltonian formalism in the Delaunay canonical variables ([18]) defined as:  $L = \sqrt{\mu a}$ ,  $G = L\sqrt{1-e^2}$ ,  $H = G \cos i$   $l = M$  mean anomaly,  $g = \omega$  argument of periapsis,  $h = \Omega$  longitude of the ascending node. The term  $(\mu/r) - (1/2)V^2$  due to the unperturbed potential, is given by:

$$H_0 = \frac{\mu^2}{2L^2} \quad (1)$$

The next step is to develop the perturbations. The zonal perturbation due to the oblateness  $J_2$  is defined by

([9]) as  $H_{20} = \varepsilon \frac{\mu}{r^3} P_{20}(\sin \phi)$  where  $\varepsilon = J_2 R^2$  ( $R = 1738 \text{ km}$  is the equatorial radius of the Moon).  $P_{nm}$  are the Legendre associated functions. Using spherical trigonometry we have  $\sin \phi = \sin i \sin(f + g)$  where  $f$  is the true anomaly and  $\phi$  the satellite position latitude (see [11] for a detailed discussion). However, the

disturbing potential is:  $H_{20} = \varepsilon \frac{\mu}{4r^3} (1 - 3 \cos^2(i) - 3 \sin^2(i) \cos(2f + 2g))$ . The terms  $\cos(f)$  and  $\sin(f)$  are substituted by known relations of the celestial mechanics ([18]) in terms of the mean anomaly. Substituting the relation  $\mu = n^2 a^3$  and with some algebraic manipulations we obtain:

$$H_{20} = \frac{15}{8} \varepsilon n^2 \left( \sin^2(i) \left( e^2 - \frac{2}{5} \right) \cos(2l + 2g) - \frac{17}{5} e^2 \sin^2(i) \cos(4l + 2g) - \frac{7}{5} e \sin^2(i) \cos(3l + 2g) + \frac{1}{5} e \sin^2(i) \cos(2g + l) - \frac{9}{5} \left( \cos^2(i) - \frac{1}{3} \right) \left( \frac{2}{9} + \frac{2}{3} e \cos(l) + \frac{1}{3} e^2 + e^2 \cos(2l) \right) \right) \quad (2)$$

For the sectorial perturbation we have ([11]):

$$H_{22} = \delta \frac{\mu}{4r^3} 3 \left\{ 2 \sin^2(i) \cos(2h) + (\cos(i) + 1)^2 \cos(2f + 2g + 2h) + (\cos(i) - 1)^2 \cos(2f + 2g - 2h) \right\}.$$

With some algebraic manipulations we get:

$$H_{22} = \delta \left( \frac{3}{4} \left( (\cos(i) - 1)^2 \cos(2l + 2g - 2h) (\cos(i) + 1)^2 \cos(2l + 2g + 2h) + 2 \sin^2(i) \cos(2h) \right) n^2 + \frac{9}{4} \left( \frac{7}{6} (\cos(i) - 1)^2 \cos(3l + 2g - 2h) - \frac{1}{6} (\cos(i) - 1)^2 \cos(l + 2g - 2h) + \frac{7}{6} (\cos(i) + 1)^2 \cos(3l + 2g + 2h) - \frac{1}{6} (\cos(i) + 1)^2 \cos(l + 2g + 2h) + \sin^2(i) (\cos(l - 2h) + \cos(2h + l)) \right) n^2 e + \frac{27}{8} \left( \frac{17}{8} (\cos(i) - 1)^2 \times \cos(4l + 2g - 2h) - \frac{5}{9} (\cos(i) - 1)^2 \cos(2l + 2g - 2h) + \frac{17}{9} (\cos(i) + 1)^2 \cos(4l + 2g + 2h) - \frac{5}{9} (\cos(i) + 1)^2 \times \cos(2l + 2g + 2h) + \left( \frac{2}{3} \cos(2h) + \cos(2l - 2h) + \cos(2l + 2h) \right) \sin^2(i) \right) n^2 e^2 \right) \quad (3)$$

where  $\delta = -C_{22}R^2$ .

### 3. THE HAMILTONIAN SYSTEM

We find in the literature several papers that use the method of the average to calculate perturbations of long period on artificial satellites of the Moon; between them we can cite [2], [7],[12], and [20-22]. However, our objective is to calculate the secular and periodic perturbations up to second order and with this, to analyze the coupling terms that appear in the application of the method. The Lie-Hori [15] perturbation method is applied for elimination of the terms of short period of the Hamiltonian.

The Hamiltonian of a satellite around the Moon can be put in the following form:

$$H = H_0 + H_1^{(0)} + H_1 + H_2$$

where

$$H_0 = \frac{\mu^2}{2L^2}; H_1^{(0)} = n_L H; H_1 = -\varepsilon H_{20}; H_2 = -\delta H_{22}$$

With the purpose of applications of the perturbation method, the terms of the Hamiltonian are put in the form:

$$H_0^{(0)} = \frac{\mu^2}{2L^2} + n_L H \tag{4}$$

$$H_1^{(1)} = H_1 + H_2$$

where the term  $n_L H$  is taken as order zero as suggested by [1] being  $n_L$  the mean motion of the Moon. The disturbing terms are represented in the first order of the applied method. The term  $n_L H$  ([14]) is added to reduce the degree of freedom since the mean longitude of the Earth time-dependent. To eliminate the terms of short period of expression (4) a method of Lie-Hori perturbation theory [15] is applied. In this work periodic terms will be calculated, substituting the result in the planetary equations of Lagrange ([16]) and finally analyzing the results. The disturbing potential in first order can be put in the form:

$$k1 = \frac{1}{8} n^2 (6\varepsilon \cos^2(i) - 3\varepsilon e^2 - 2\varepsilon - 18\delta \cos(2h)e^2 + 18\delta \cos(2h)e^2 \cos^2(i) - 12\delta \cos(2h) + 12\delta \cos(2h) \cos^2(i) + 9\varepsilon \cos^2(i)e^2) \tag{5}$$

where  $\varepsilon = J_2 R^2$  e  $\delta = -C_{22} R^2$

We observe that at this considered order of the disturbing potential the terms due to oblateness ( $\varepsilon$ ) are secular terms whereas, the terms of the equatorial ellipticity of the Moon ( $\delta$ ) appear multiplied for  $\cos(2h)$ . The second order disturbing potential (order of the application of the Hori method) is very extensive and it won't be exposed here. The terms due to the oblateness and the equatorial ellipticity of the Moon appear in the potential of following form  $\varepsilon^2$ ,  $\delta^2$ ,  $\varepsilon\delta$ , terms of coupling between the  $J_2$  and the  $C_{22}$  and terms of second order  $J_2^2$  and  $C_{22}^2$ . In this case, the terms of the oblateness ( $\varepsilon$ ) also appear multiplied for  $...\varepsilon \cos(2h - 2g) + ... \varepsilon \cos(2g) + ...$  among others terms.

### 4. CRITICAL INCLINATION

Taking into account the first order disturbing potential and considering in the Hamiltonian the effects of the  $J_2$  and  $C_{22}$  a new formula for the critical inclination is found. In fact, solving the equation  $dw/dt = 0$ , we get

$$\cos^2(i) = \frac{\varepsilon + \delta \cos(h)}{5(\varepsilon + 2\delta \cos(h))} \tag{6}$$

When we consider the term due the equatorial ellipticity of the Moon the critical inclination depends on the longitude of the ascending node. Figure 1 represents the variation between the inclination and the ascending node.

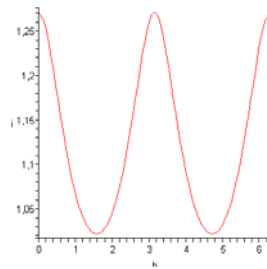


Figure 1: Variation of the critical inclination in relation the longitude of the ascending node.

Table 2 represents the values of the critical inclination for some values of the ascending node.

Table 2: Critical inclination for the potential of first order, where:  $\epsilon = 613.573$ ;  $\delta = -67.496$

Longitude of the ascending node ( $h$ )	Critical inclination for direct orbits ( $i_c$ )	Critical inclination for retrograde orbits ( $i_c$ )
1 rad	61.10°	118.90°
2 rad	59.98°	120.02°
$\pi/2$	58.56°	121.45°
$\pi/3$	60.69°	119.31°
$\pi$	72.83°	107.17°

Now let us consider the disturbing potential of second order. Substituting in the planetary equations of Lagrange and solving the equation  $dw/dt=0$ , we get a formula to compute the critical inclination. The equation won't be exposed here since they are very extensive and is a function of two variables that are: the argument of the periaapsis ( $g$ ) and the longitude of the ascending node ( $h$ ). The Figures 2 and 3 represent the variation of the critical inclination fixing one of the involved variables. The Table 3 represents the critical inclination taking into account the second order disturbing potential and considering some values for the argument of the periaapsis and the ascending node. When the sectorial term  $C_{22}$  is considered, the first order disturbing potential is function of the longitude of the ascending node and of both ascending node and argument of the periaapsis in the case of the second order potential. Considering  $g$  and  $h$  varying, we can get the variation for the critical inclination as can be seen in Figure 4.

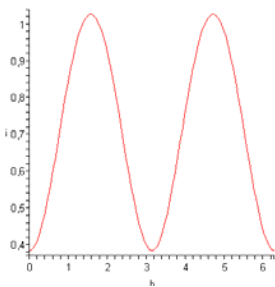


Figure 2: Variation of the critical inclination with fixed  $g$ .

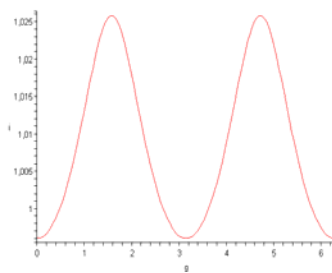


Figure 3: Variation of the critical inclination with fixed  $h$ .

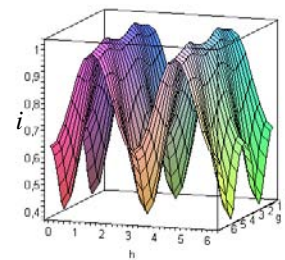


Figure 4: Variation of the critical inclination with  $g$  and  $h$  varying.

Table 3: Critical inclination for the potential of second order

Argument of periapsis ( $g$ )	Longitude of the ascending node ( $h$ )	Critical inclination, for direct orbits ( $i_c$ )	Critical inclination, retrograde orbit ( $i_c$ )
$\pi/2$	$2 \text{ rad}$	$33.29^\circ$	$146.29^\circ$
$1 \text{ rad}$	$3\pi/2$	$38.88^\circ$	$140.86^\circ$
$2 \text{ rad}$	$\pi/2$	$38.61^\circ$	$141.12^\circ$
$2 \text{ rad}$	$1 \text{ rad}$	$28.63^\circ$	$148.40^\circ$
$\pi$	$2\pi/3$	$34.16^\circ$	$145.48^\circ$

## 5. APPLICATIONS FOR SATELLITES OF LOW ALTITUDE

The disturbing potential is substituted in the Lagrange's planetary equations ([16]) and integrated numerically. Considering the disturbing potential due the non-sphericity of the Moon ( $J_2$  and  $C_{22}$ ) numerical applications are done to analyze the variation of the eccentricity and inclination for different initial conditions of the inclination.

Figures 5 and 6 represent the comparison for different orders of the disturbing potential. The variation of the eccentricity for lunar satellites of low altitude is constant at first order. This happens because the coefficients  $J_2$  and  $C_{22}$  don't affect the variation rate of the eccentricity (as we can verify in the Lagrange's planetary equations). Therefore, is important to insert more terms in the potential to get more realistic results, as for example, to study the lifetimes of low altitude Moon artificial satellites [17] considering the zonal terms  $J_2$ ,  $J_3$  and  $J_5$  and the sectorial terms  $C_{22}$  and  $C_{31}$  in the disturbing potential. For the second order presents a slow increase in its variation rate. This small variation is due to the fact that the given initial conditions are for frozen orbits ([5-6]). The expression of the eccentricity is presented in the following form  $de/dt = \dots \sin(2g) + \dots \sin(2g + 2h) \dots - \dots \sin(2g - 2h) \dots + \dots$ . Another factor that also contributes for the small variation of the eccentricity, using the potential of second order, is because the coefficients of second order ( $J_2^2$ ,  $C_{22}^2$ ) and the coupling terms between  $J_2$  and  $C_{22}$ , and due to difference between the mean motion of the Moon ( $n_L$ ) and the satellite ( $n$ ) that appear in the denominator of the equations ( $1/(n - n_L)$ ).

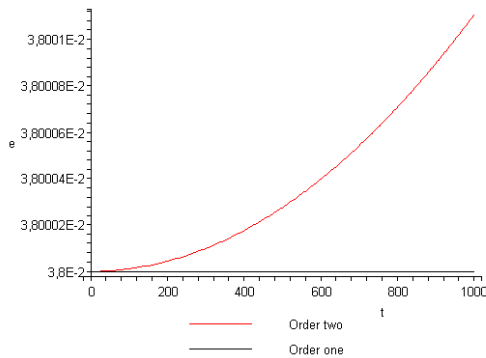


Figure 5: Initial conditions:  $a= 1838 \text{ km}$ ,  $e=0.03$ ,  $i=30^\circ$ ,  $g = 3\pi/2$ ,  $h = \pi/2$  and t-days,  $i$ -rad.

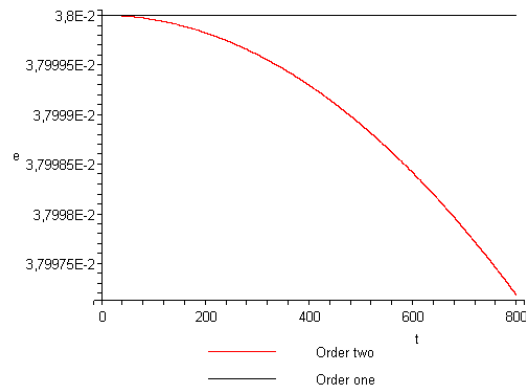


Figure 6: Initial conditions:  $a= 1838 \text{ km}$ ,  $e=0.03$ ,  $i=65^\circ$ ,  $g = 3\pi/2$ ,  $h = \pi/2$  and t-days,  $i$ -rad.

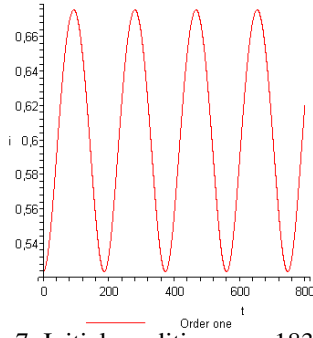


Figure 7: Initial conditions:  $a= 1838$  km,  $e=0.03$ ,  $i=30^\circ$ ,  $g = 3\pi/2$ ,  $h = \pi/2$  and t-days,  $i$ -rad.

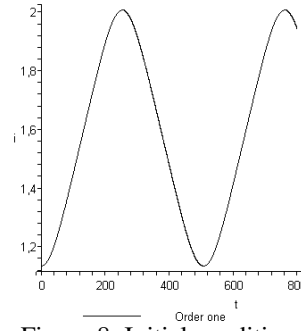


Figure 8: Initial conditions:  $a= 1838$  km,  $e=0.03$ ,  $i=65^\circ$ ,  $g = 3\pi/2$ ,  $h = \pi/2$  and t-days,  $i$ -rad.

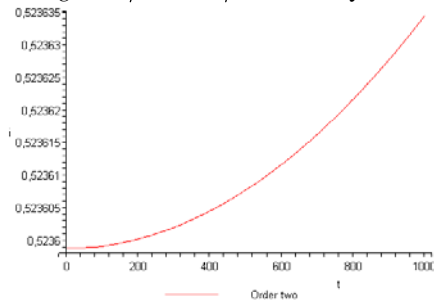


Figure 9: Initial conditions:  $a= 1838$  km,  $e=0.03$ ,  $i=30^\circ$ ,  $g = 3\pi/2$ ,  $h = \pi/2$  and t-days,  $i$ -rad.

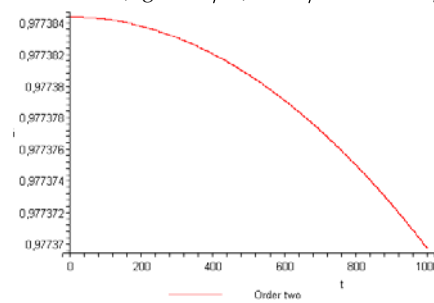


Figure 10: Initial conditions:  $a= 1838$  km,  $e=0.03$ ,  $i=65^\circ$ ,  $g = 3\pi/2$ ,  $h = \pi/2$  and t-days,  $i$ -rad.

Figure 7 and 8 shows that the inclination suffers a periodic variation due to the first order because of the longitude of the ascending node, as we can verify by equation (6). Figure 7 shows a slow variation of the inclination for a value below of the critical inclination while Figure 8 presents a variation more accentuated for a value of the inclination above of the critical inclination. For the second order, Figure 9 and 10 represent the variation rate of the inclination that is very small due to the initial values given for the  $\omega$  argument of the periaapsis and for the longitude of the ascending node. The equation for the inclination,  $di/dt = \dots \sin(2g) + \dots \sin(2g + 2h) \dots + \dots$  depends on the sine of these arguments. The given initial values for  $g$  and  $h$  are due the condition of frozen orbits ([5]). The same comments done for the eccentricity, including those for the mean motions, coupling terms (zonal and sectorial) too are valid for the variation of the orbital inclination.

## 6. CONCLUSION

Using Lie-Hori method the disturbing potential due the non-sphericity of the gravitational field of the Moon in first and second order is obtained. The disturbing potential is substituted in the Lagrange's planetary equations and integrated numerically. Analyses with the variations of the orbital elements are done. In the second order potential terms of couplings between the oblateness and the equatorial ellipticity of the Moon ( $J_2 C_{22}$ ) and terms of second order of type  $J_2^2$  and  $C_{22}^2$  are obtained. A formula is developed to compute the critical inclination when the effect of the  $C_{22}$  (equatorial ellipticity) term is considered in the Hamiltonian in first and second order. The critical inclination can be strongly affected by the coefficient due to equatorial ellipticity of the Moon and by the value of the longitude of the ascending node. The formula of the critical inclination for the second order is a function of two variables: the argument of the periaapsis and the longitude of the ascending node. At the first order this formula is a function of the longitude of the ascending node only. For artificial satellites of the Moon of low altitude it is important to take in account the terms due to the oblateness and the equatorial ellipticity of the Moon to get resulted next to the real.

When it is considered only the non-sphericity of the Moon in the disturbing potential for a lunar satellite of low altitude, at first order, the eccentricity orbital of the satellite is constant along the time but for at the second order it presents small variations. In the case of the first order it is due to the fact that the

coefficients  $J_2$  and  $C_{22}$  don't affect the variation rate of the eccentricity (as we can verify in the Lagrange's planetary equations), therefore it is important to insert more terms in the potential to get resulted still more realistic, as for example, the zonal terms  $J_3$ ,  $J_4$  and  $J_5$  and the sectorial term  $C_{31}$ . Now, for the second order, the small variation of the eccentricity is due to the fact that the initial conditions are given to get frozen orbits and also because the terms appearing in the denominator of the equations.

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