The thermodynamic problem with moving boundary

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Resumo: *A mathematical model of the linear thermodynamic equations with moving ends, based on the Stefan Problem is considered. In this work, we are interested in obtaining existence, uniqueness and regularity using the Faedo-Galerkin method. For numerical solutions, we shall employ the finite element method together with the Crank-Nicolson method. A numerical experiment is presented to show the moving boundary for the problem.*

Palavras-chave: *Termal Problem; Crank-Nicolson method; Moving Boundary; Finite Element Method.*

1 Introduction

The thermal equations with moving ends give origin to several physical problems, such as, Stefan Problem, who can be interpreted as flame propagation problem, study of atom movements, study of combustion theory. Flame propagation problems have been established in [2] that studied a mathematical formulation suitable in the analyze for the theory of diffusion flames. Studies the mathematical model of moving ends arising in combustion theory was developed by Schmidt-Lainé [1] and [3]. The discharge or absorption of thermal energy during the change in atom arrangement have been studied in [5]. We can also see the problem as a model for certain processes involving chemical reactions. The objective of this paper is to obtain existence and uniqueness of solutions and too we apply the finite element method with a finite difference method in time to obtain an approximated numerical solution.

2 Formulation Problem

Our objective is study the approximated solutions of the following thermodynamics system,

$$(I) \quad \begin{cases} \dfrac{\partial u}{\partial t} - k\dfrac{\partial^2 u}{\partial x^2} = f(x,t), & (x,t) \in Q_t \\ u(\alpha(t),t) = 0, \quad u(\beta(t),t) = 0, & \forall x \in (\alpha(t),\beta(t)), \quad \forall t \in [0,T] \\ u(x,0) = u_0(x), & \alpha(0) < x < \beta(0) \end{cases} \tag{1}$$

where $k > 0$ is the thermal conductivity.

We consider the domain $Q_t \subset I\!R^2$ defined by $Q_t = \Big\{(x,t) \in I\!R \times (0,T)\,;\, \alpha(t) < x < \beta(t)\Big\}$ and the horizontal length of the interval is defined by $\gamma(t) = \beta(t) - \alpha(t) > 0$.

Let us study the existence, uniqueness, and the approximated solution of the Problem (1). For this we consider the following hypothesis:

H1: $\alpha, \beta \in C^2(0,\infty)$ in $0 < \gamma_0 < \gamma(t) < \gamma_1$, $\forall t \geq 0$,

H2: $\alpha', \beta' \in L^\infty(0,\infty)$, $\forall t \geq 0$

where γ_0 and γ_1 are positive constants. Consider now the change of variable to transform the domain Q_t in a cylindric domain Q, given by

$$\tau : Q_t \longrightarrow Q = (0,1) \times]0,T[, \qquad (y,t) = \Big(\frac{x-\alpha(t)}{\gamma(t)}, t\Big). \tag{2}$$

Note that $\tau \in C^2(Q_t)$ and by inverse function Theorem, the application τ^{-1} also is $C^2(Q)$.

The change of variable $v(y,t) = u(\alpha(t) + \gamma(t)y, t)$, implies that $u(x,t) = v\Big(\frac{x-\alpha(t)}{\gamma(t)}, t\Big)$. Using the application τ, we obtain the following cylindrical problem:

$$\text{(II)} \quad \begin{cases} v' - a(t)\dfrac{\partial^2 v}{\partial y^2} - b(y,t)\dfrac{\partial v}{\partial y} = g(y,t), & \text{in } Q \\ v(0,t) = 0, \quad v(1,t) = 0, \quad t \geq 0 \\ v(y,0) = v_0(y), \qquad 0 < y < 1. \end{cases} \tag{3}$$

where $a(t) = k/\gamma^2$, $b(y,t) = (\alpha' + \gamma' y)/\gamma$ We shall show the existence and uniqueness from the solution of the cylindrical Problem (3) who implies in the existence and uniqueness of the solution of the noncylindrical Problem (1), asymptotic behavior and numerical results.

3 Existence and Uniqueness

We will investigate global existence and uniqueness solution for Problem(3) and consequently for Problem (1). For this, consider the notation $\Omega = (0,1)$, $\Omega_t = (\alpha(t), \beta(t))$ and $\Omega_0 = (\alpha(0), \beta(0))$.

Theorem 1 *Under the hypotheses* (**H1**) *and* (**H2**) *and the initial data* $v_0 \in H_0^1(\Omega)$ *and* $g \in L^1(0,\infty; L^2(\Omega)) \cap L^2(0,T; L^2(\Omega))$*, there exists only one strong solution* $v : Q \longrightarrow I\!R$ *of the problem (II), that is,*

$$v' - a(t)\frac{\partial^2 v}{\partial y^2} - b(y,t)\frac{\partial v}{\partial y} = g(y,t) \qquad in \quad L^2(0,T; L^2(\Omega)),$$

satisfying the following conditions:
(i) $v \in L^2(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega))$, *(ii)*$v' \in L^2(0,T; L^2(\Omega))$.

And consequently we obtain the following result,

Theorem 2 *Under the hypotheses* (**H1**) *and* (**H2**) *and the initial data* $u_0 \in H_0^1(\Omega_0)$ *and* $f \in L^1(0,\infty; L^2(\Omega_t)) \cap L^2(0,\infty; L^2(\Omega_t))$ *, there exists only one strong solution* $u : Q_t \longrightarrow I\!R$ *of the problem* (I)*, that is,*

$$u' - k\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad in \quad L^2(0,\infty; L^2(\Omega_t))$$

satisfying the following conditions:
(i) $u \in L^\infty(0,\infty; H_0^1(\Omega_t) \cap H^2(\Omega_t))$, *(ii)*$u' \in L^2(0,\infty; L^2(\Omega_t))$.

Proof of the Theorem1: We introduce the approximate solutions. Let $T > 0$ and denote by V_m the subspace spanned by $\{w_1, w_2, ..., w_m\}$, where $\{w_\nu; \nu = 1, \cdots m\}$ are the first m eigenvectors of the space $H_0^1(\Omega)$, solution of the spectral problem -$\dfrac{\partial^2 w_i}{\partial x^2} = \lambda_i w_i$. If $v_m(t) \in V_m$ then it can be represented by

$$v_m(y,t) = \sum_{\nu=1}^{m} g_{\nu m}(t) w_\nu(y),$$

where v_m is the solution of the system of ordinary differential equations

$$\begin{cases} (v'_m, w) + a(t)\Big(\dfrac{\partial v_m}{\partial y}, \dfrac{\partial w}{\partial y}\Big) - \Big(b(y,t)\dfrac{\partial v_m}{\partial y}, w\Big) = (g, w), \quad \forall w \in V_m \\ v_m(0) = v_{0m} \longrightarrow v_0 \quad \text{in} \quad H_0^1(\Omega), \end{cases} \tag{4}$$

The system (4) has local solution in the interval $(0, T_m)$. To extend the local solution to the interval $(0, T)$ independent of m the following a priori estimate is needed.

Estimate I

Taking $w = v_m \in V_m$ in (4), we obtain

$$(v'_m, v_m) + a(t)\Big(\frac{\partial v_m}{\partial y}, \frac{\partial v_m}{\partial y}\Big) - \Big(b(y,t)\frac{\partial v_m}{\partial y}, v_m\Big) = (g, v_m). \tag{5}$$

Integrating the third term of (5) by parts, using the boundary conditions and the norma equivalence of norms in H_0^1, and after Integrating from $[0,t)$, with $t \in [0, T_m)$, we get,

$$\frac{1}{2}|v_m(t)|^2 + a(t)\int_0^t \|v_m\|^2 \;\; \le \;\; \frac{1}{2}|v_m(0)|^2 + \frac{1}{2}|g|^2 + \int_0^t \frac{1}{2}\Big(1 + |\frac{\gamma'}{\gamma}|\Big)|v_m|^2.$$

From the hypotheses (H1) and (H2), $\gamma' \in L^1(0,\infty)$, then

$$\int_0^t \frac{1}{2}\Big(1 + |\frac{\gamma'}{\gamma}|\Big)|v_m|^2 \le \int_0^t \frac{1}{2}\Big(1 + |\frac{\gamma'}{\gamma_0}|\Big)|v_m|^2 \;\; \le \;\; c\int_0^t |v_m|^2.$$

where c is a positive constant. Let $c_1 = \min\{\frac{1}{2}, a(t)\}$, then

$$|v_m(t)|^2 + \int_0^t \|v_m\|^2 \;\; \le \;\; \frac{1}{2c_1}|v_m(0)|^2 + \frac{1}{2c_1}|g|^2 + \frac{c}{c_1}\int_0^T |v_m|^2.$$

By the use of Gronwall's Lemma in the last inequality, we obtain the following estimates

$$v_m \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)) \quad \textit{and} \quad v_m \quad \text{is bounded in} \quad L^2(0,T;H_0^1(\Omega)). \tag{6}$$

Estimate II

Taking $w = v'_m$ in (4), we obtain

$$(v'_m, v'_m) + a(t)\Big(\frac{\partial v_m}{\partial y}, \frac{\partial v'_m}{\partial y}\Big) - \Big(b(y,t)\frac{\partial v_m}{\partial y}, v'_m\Big) = (g, v'_m), \quad \forall v'_m \in V_m. \tag{7}$$

The second term on left hand side in (7) implies that

$$a(t)\Big(\frac{\partial v_m}{\partial y}, \frac{\partial v'_m}{\partial y}\Big) = \frac{a(t)}{2}\frac{d}{dt}\|v_m\|^2, \tag{8}$$

by equivalence of norms in $H_0^1(\Omega)$. For the third term in (7) can be estimated as,

$$b(y,t)\Big|\frac{\partial v_m}{\partial y}\Big|\;|v'_m| \le \frac{|\alpha'| + |\gamma'|}{\gamma_0}\|v_m\|\;\;|v'_m|. \tag{9}$$

Substituting (8),(9) in the equation (7), we get

$$|v'_m|^2 + \frac{a(t)}{2}\frac{d}{dt}\|v_m\|^2 \le \frac{|\alpha'| + |\gamma'|}{\gamma_0}\|v_m\||v'_m| + |g|\;|v'_m|. \tag{10}$$

From elementary inequality

$$\frac{|\alpha'|+|\gamma'|}{\gamma_0}\|v_m\|\;|v_m'|+|g|\;|v_m'|\leq 4\Big(\frac{|\alpha'|^2+|\gamma'|^2}{|\gamma_0|^2}\Big)\|v_m\|^2+2|g|^2+\frac{1}{2}|v_m'|^2.$$

Therefore,

$$\frac{1}{2}|v_m'|^2+\frac{a(t)}{2}\frac{d}{dt}\|v_m\|^2\ \leq\ 2\Big(\frac{|\alpha'|^2+|\gamma'|^2}{|\gamma_0|^2}\Big)\|v_m\|^2+2|g|^2.$$

We observe that, from the hypothesis (H1), $\gamma(t)<\gamma_1,\ \forall t\in[0,T]$.

Integrating in $0<t<T_m$, using the hypothesis (H1) and the definition $a(t)$, we obtain

$$\frac{1}{2}\int_0^t|v_m'|^2+\frac{k}{2\gamma_1^2}\int_0^t\frac{d}{dt}\|v_m\|^2\ \leq\ 4\int_0^T\frac{|\alpha'|^2+|\gamma'|^2}{|\gamma|^2}\|v_m\|^2+2\int_0^T|g|^2. \tag{11}$$

Note that, from hypothesis (H2), there is a positive constant $\hat{c}$, such that

$$4\int_0^T\frac{|\alpha'|^2+|\gamma'|^2}{|\gamma|^2}\|v_m\|^2\leq\widehat{c}\int_0^T\|v_m\|^2,$$

Then, substituting in (11) and using the Gronwall inequality we have

$$v_m\quad\text{is bounded in}\quad L^\infty(0,T;H_0^1(\Omega))\quad \textit{and}\quad v_m'\quad\text{is bounded in}\quad L^2(0,T;L^2(\Omega)). \tag{12}$$

Estimate III

Taking $w=-\dfrac{\partial^2 v_m}{\partial y^2}$ in (4) and after to integrate the second term by parts we have

$$-\Big(v_m',\frac{\partial^2 v_m}{\partial y^2}\Big)+a(t)\Big(\frac{\partial^2 v_m}{\partial y^2},\frac{\partial^2 v_m}{\partial y^2}\Big)+\Big(b(y,t)\frac{\partial v_m}{\partial y},\frac{\partial^2 v_m}{\partial y^2}\Big)=-\Big(g,\frac{\partial^2 v_m}{\partial y^2}\Big). \tag{13}$$

The first and the second term on the left hand side in (13) implies that

$$-\Big(v_m',\frac{\partial^2 v_m}{\partial y^2}\Big)=\int_0^1\frac{\partial v_m'}{\partial y}\frac{\partial v_m}{\partial y}\frac{1}{2}\frac{d}{dt}\|v_m\|^2,\qquad a(t)\Big(\frac{\partial^2 v_m}{\partial y^2},\frac{\partial^2 v_m}{\partial y^2}\Big)\geq\frac{k}{\gamma_1^2}\Big|\frac{\partial^2 v_m}{\partial y^2}\Big|^2. \tag{14}$$

As $0<y<1$ and from the hypothesis (H1), we have

$$\Big(\frac{|\alpha'|+|\gamma'|}{\gamma_0}\Big)\|v_m\|\Big|\frac{\partial^2 v_m}{\partial y^2}\Big|\leq\frac{4\gamma_1^2}{k}\Big(\frac{|\alpha'|+|\gamma'|}{\gamma_0}\Big)^2\|v_m\|^2+\frac{k}{4\gamma_1^2}\Big|\frac{\partial^2 v_m}{\partial y^2}\Big|^2. \tag{15}$$

Similarly we have

$$-\Big(g,\frac{\partial^2 v_m}{\partial y}\Big)\leq\Big|g\Big|\;\Big|\frac{\partial^2 v_m}{\partial y^2}\Big|\leq\frac{4\gamma_1^2}{k}\Big|g\Big|^2+\frac{k}{4\gamma_1^2}\Big|\frac{\partial^2 v_m}{\partial y^2}\Big|^2. \tag{16}$$

Substituting (14), (15) and (16) in the equation (13), we obtain

$$\frac{d}{dt}||v_m||^2+\frac{k}{\gamma_1^2}\Big|\frac{\partial^2 v_m}{\partial y^2}\Big|^2\leq 2C_1||v_m||^2+\frac{8\gamma_1^2}{k}|g|^2\leq C(||v_m||^2+|g|^2).$$

where

$$C_1=\frac{4\gamma_1^2}{k}\Big(\frac{|\alpha'|+|\gamma'|}{\gamma_0}\Big)^2,\qquad C=\max\Big\{2C_1,\frac{8\gamma_1^2}{k}\Big\}$$

Integrating from 0 to $t<T_m$, using the Gronwall inequality we conclude that

$$v_m\quad\text{is bounded in}\quad L^\infty(0,T;H_0^1(\Omega))\quad \textit{and}\quad \frac{\partial^2 v_m}{\partial y^2}\quad\text{is bounded in}\quad L^2(0,T;L^2(\Omega)). \tag{17}$$

The estimates obtained in (6), (12) and (17), permit us to pass the limits in the approximate system (4) in the Faedo-Galerkin method and hence, we have proved the existence of solutions $v(y,t)$ in the sense defined in Theorem 1.

Uniqueness: The uniqueness of solution is made by contradiction and does not prove that work.□

Proof of the Theorem2: Let v the solution of the Problem (3), with the following initial data $v_0(y) = u_0(\alpha(0) + \gamma(0)y)$. Consider the function $u(x,t) = v(y,t)$, where $x = \alpha(t) + \gamma(t)y$. To verify that $u(x,t)$, under the hypotheses of Theorem 2, is the solution of Problem 1, it is sufficient to observe that the mapping $\tau : Q_t \longrightarrow Q$ is of class C^2. Since the Problems 1 and 3 are equivalently, then u satisfy the Problem 1.□

4 Approximate Solution

Our goal in this section is the numerical implementation of approximate solutions. To obtain the numerical approximate solutions we will use both finite element method and finite difference method. Moreover, some numerical experiments will be presented to analyze the effect of the moving boundary in the thermodynamics system (1).

Let V_m the subspace spanned by $\{\varphi_1, \varphi_2, \ldots, \varphi_{m+1}\}$, where $\{\varphi_\nu;\ \nu = 1, \cdots, m+1\}$ are the first $m+1$ bases vectors of the space $V = H_0^1(\Omega)$. If $v^h(y,t) \in V_m$, then it can be represented by

$$v^h(y,t) = \sum_{i=1}^{m+1} c_i(t)\varphi_i(y), \qquad \varphi_i(y) \in V_m. \tag{18}$$

Restricting the equation (3) in the subspace V_m and taking v^h in (18) and, $w = \varphi_j(y)$, we find that

$$\begin{aligned}\sum_{i=1}^{m+1} \Big\{ c_i'(t) \int_0^1 \varphi_i(y)\varphi_j(y)dy \,+ c_i(t)a(t) \int_0^1 \frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}dy \\ -c_i(t) \int_0^1 b(y,t)\frac{\partial\varphi_i}{\partial y}\varphi_j dy \Big\} = \int_0^1 g(y,t)\varphi_j(y)\, dy,\end{aligned} \tag{19}$$

We define

$$A = \int_0^1 \frac{\partial\varphi_i(y)}{\partial y}\frac{\partial\varphi_j(y)}{\partial y}\, dy, \qquad B(t) = \int_0^1 b(y,t)\frac{\partial\varphi_i(y)}{\partial y}\varphi_j(y)\, dy$$

$$D = \int_0^1 \varphi_i(y)\varphi_j(y)\, dy, \qquad G(t) = \int_0^1 g(y,t)\varphi_j(y)\, dy,$$

where $A, B(t)$ and D are $m \times m$ square matrix and G is a vector known $m \times 1$. Substituting the matrices in (19), we obtain the following ordinary differential system,

$$Dc'(t) + \Big(a(t)A - B(t)\Big)c(t) = G(t), \qquad \forall t \geq 0 \tag{20}$$

4.1 Finite Difference Method and Finite Element Approximation

In order to solve the system (20) in each discrete time, we will apply the numerical method due to Crank-Nicolson (see,Hugles [4]). Setting $t = t_n$ in (20), we obtain the iterative method

$$\Big(D + \frac{\Delta t}{2}(a^n A - B^n)\Big)c^{n+1} = \Big(D - \frac{\Delta t}{2}(a^n A - B^n)\Big)c^n + \frac{\Delta t}{2}\Big(G^{n+1} + G^n\Big), \quad \text{for} \quad n = 0, 1, \cdots, N. \tag{21}$$

Suppose that the matrices, $D = D_{ij}$, $A = A_{ij}$ and $B = B_{ij}$ are known, then the iterative method (21) can be easily implemented. Taking $t = 0$ into (21) yields,

$$\Big(D + \frac{\Delta t}{2}(a^0 A - B^0)\Big)c^1 = \Big(D - \frac{\Delta t}{2}(a^0 A - B^0)\Big)c^0 + \frac{\Delta t}{2}\Big(G^1 + G^0\Big). \tag{22}$$

To calculate the matrices of linear system (21), we need to introduce the basis function $\varphi_i \in V_m$. More specifically, in this work, we have used the hat function, i.e, piecewise linear functions, as the basis function of V_m subspace, considering the uniform mesh, $h = h_i = y_{i+1} - y_i$, $i = 1, 2, \ldots, m$ in the discretization.

5 Numerical Simulation

In this section, attention is turned to implementation of approximate solutions. A numerical experiments will be given to analyze the effect of the moving ends of Ω_t. For this, we introduce an appropriate external force $f(x,t)$ sufficiently regular on the right-hand side of the original Problem (1). Therefore, the exact solution is known for suitable choice of $f(x,t)$, which allow us to get the numerical simulation for Problem(1) or Problem (3). The error estimates, in the semi-discrete or fully discrete, can be found [6]. This will be the procedure for analysis of the convergence.

Example Let $v(y,t) = \dfrac{1}{\pi^2} sin(\pi y) cos(\pi t)$ be a exact solution of the o Problem (3) for suitable choice of $g(y,t)$. Then, the initial and boundary conditions read, respectively $v(y,0) = \dfrac{1}{\pi^2} sin(\pi y)$, $\quad v(0,t) = v(1,t) = 0$. Considering the ends functions of Ω_t define by

$$\alpha(t) = -\frac{t}{t+1} \quad \text{and} \quad \beta(t) = \frac{2t+1}{t+1}, \quad \text{then} \quad \gamma(t) = \beta(t) - \alpha(t) = \frac{3t+1}{t+1}.$$

Numerical Error In the norm $L^\infty(0,T;L^2(\Omega_t))$ the numerical error is calculated by

$$E_{L^\infty(0,T;L^2(\Omega_t))} = \max_{t_n \in [0,1]} \Big(\int_{\alpha(t)}^{\beta(t)} |u(x_i,t_n) - u_h(x_i,t_n)|^2 dx \Big)^{1/2}.$$

The table that follow, is shown the errors obtained between exact and approximate solutions to the problem (1) when the step time is fixed $\Delta t = 0.01$ and for several sizes of mesh $h = 0.1; 0.05; 0.02; 0.01; 0.001$ that represent respectively $m = 10; 20; 50; 100; 1000$ nodes of the interval $[\alpha(t),\ \beta(t)]$. For this case were made 100 iterations in the iterative method (21).

Table

Δt	h	$E_{L^\infty(0,T;L^2(\Omega_t))}$
0.01	0.1	0.005747
0.01	0.05	0.005723
0.01	0.02	0.005717
0.01	0.01	0.005688
0.01	0.001	0.005652

Asymptotic behavior of the energy The goal in this section is to establish a rate decay for the energy of system (1). Therefore, the asymptotic behavior, as $t \to \infty$, of the natural energy

$$E(t) = \frac{1}{2} |u(t)|^2_{L^2(\Omega_t)} \tag{23}$$

will be obtained inside of the time dependent domain Q_t. Thus, we can state

Theorem 3 *Assuming the hypotheses of Theorem 2, then the energy $E(t)$ associated with the global weak solution of system (1) satisfies*

$$E(t) \leq E(0) e^{-2tk/\gamma_1^2} \qquad \forall t \geq 0.$$

where k is thermal conductivity and γ_1 is a positive real constant given in **(H1)**.

In order to prove the Theorem 4 one needs to establish the following Poincarè inequality in Ω_t,

Lemma 1 *If $u \in H_0^1(\Omega_t)$ then the Poicaré inequality*

$$|u(t)|^2_{L^2(\Omega_t)} \leq \gamma^2(t)|u_x(t)|^2_{L^2(\Omega_t)}. \tag{24}$$

Proof of Theorem 3 - Multiplying both sides of $(1)_1$ by $u(x,t)$ and integrating on Ω_t yields

$$\int_{\alpha(t)}^{\beta(t)} \Big[u_t(x,t) - ku_{xx}(x,t)\Big] u\,(x,t)\,dx = 0. \tag{25}$$

Applying the Leibnitz rule in each term of (25) one has

$$\int_{\alpha(t)}^{\beta(t)} u\,(x,t)\,u_t\,(x,t)\,dx = \frac{1}{2}\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} |u\,(x,t)|^2\,dx - \frac{1}{2}u^2\,(\beta\,(t)\,,t)\,\beta'\,(t) + \frac{1}{2}u^2\,(\alpha\,(t)\,,t)\,\alpha'\,(t)\,.$$

From this and boundary conditions imply that

$$\int_{\alpha(t)}^{\beta(t)} u\,(x,t)\,u_t\,(x,t)\,dx = \frac{1}{2}\frac{d}{dt}\,|u\,(t)|^2_{L^2(\Omega_t)}\,. \tag{26}$$

Integrating by parts and using the boundary conditions, in the second term yield

$$\int_{\alpha(t)}^{\beta(t)} u(x,t)u_{xx}(x,t)dx = |u_x(t)|^2_{L^2(\Omega_t)}. \tag{27}$$

Inserting (26) and (27) in (25) and using the Lemma 1 and hypothesis **(H1)** one gets

$$|u\,(t)|^2_{L^2(\Omega_t)} \leq \gamma_1^2\,|u_x\,(t)|^2_{L^2(\Omega_t)}\,.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\,|u\,(t)|^2_{L^2(\Omega_t)} \leq -\frac{k}{\gamma_1^2}\,|u\,(t)|^2_{L^2(\Omega_t)} \quad \text{for all} \quad t \geq 0. \tag{28}$$

From (23) and (28) one gets

$$\frac{d}{dt}\left\{e^{2kt/\gamma_1^2}E\,(t)\right\} \leq 0. \tag{29}$$

Integrating from 0 to t we conclude the demonstration of Theorem 3 □

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