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On Persistent Centers

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RESUMO

The problem of distinguishing whether a monodromic critical point with imaginary eigenvalues of a family of planar analytical vector field is a center or a focus was already solved by Lyapunov. This problem is usually called *the center-focus problem.* The method consists in computing several quantities associated to the point, the so called *Lyapunov constants*, and study whether they are zero or not. Without the aim of being exhaustive we quote some different methods addressed to compute them, collected according the approaches that they use: computation of a Lyapunov function, following the first method introduced by Lyapunov; use of normal forms; computation of the power expansion of a solution of the system in polar coordinates; use of the algebraic structure of the Lyapunov constants; method of Lyapunov-Schmidt; method of averaging, ...

In despite of all the above results the solution of the center focus-problem for natural and simple families, like for instance the cubic systems, has resisted all the attempts. For this reason in this paper we propose to grade the centers in three levels in order to make the problem more feasible. Before introducing our results recall that a weak focus of a real analytic planar autonomous differential equation can always we written as

$$\dot{z} = iz + F(z,\bar{z}) = iz + \sum_{k=2}^{\infty} F_k(z,\bar{z}), \tag{1}$$

where $z = x + iy$ and F_k are complex homogeneous polynomials of degree k easily constructed from the corresponding P_k and Q_k. In this paper we will always work with this second equivalent expression.

Definition 1. *Given a differential equation* (1), $\dot{z} = iz + F(z,\bar{z})$, *we say that:*

- *The origin is a* **persistent center** *if it is a center for*

$$\dot{z} = iz + \lambda F(z,\bar{z})$$

  for all $\lambda \in \mathbb{C}$.

- *The origin is a* **weakly persistent center** *if it is a center for*

$$\dot{z} = iz + uF(z,\bar{z})$$

  for all $u \in \mathbb{R}$.

Notice that clearly

$$\{\text{persistent centers}\} \subset \{\text{weakly persistent centers}\} \subset \{\text{centers}\}.$$

This paper deals with the smallest category, the persistent centers. A similar concept could also be introduced regarding to degenerate centers. The weakly persistent centers are not studied in this work. We remark that this class forms a much more extensive family of centers. For instance, in case of equations with homogeneous non-linearities, it can be easily seen that they coincide with all the centers.

Our first result is a catalog of all the persistent centers that we have found.

Theorem 0.1. *The origin of system* (1) *is a persistent center in the following cases:*

(a) $\dot{z} = iz + Az^2 + C\bar{z}^2$ *(quadratic),*

(b) $\dot{z} = iz + f(z)$, *with* $f(0) = f'(0) = 0$ *(holomorphic),*

(c) $\dot{z} = iz + f(\bar{z})$, *with* $f(0) = f'(0) = 0$ *(hamiltonian),*

(d) $\dot{z} = iz + z\bar{z}f(\bar{z})$ *(separated).*

(e) $\dot{z} = iz + Bz^k\bar{z}^l\psi(z\bar{z})$ *with* $k \neq l + 1$ *(reversible),*

where f *is a complex analytic function,* $A, B, C \in \mathbb{C}$ *and* ψ *is a real analytic function such that* $z^k\bar{z}^l\psi(z\bar{z})$ *starts at least with second order terms.*

Theorem 0.2. *Consider the complex differential equation*

$$\dot{z} = iz + Az^2 + Bz\bar{z} + C\bar{z}^2 + Dz^3 + Ez^2\bar{z} + Fz\bar{z}^2 + G\bar{z}^3. \tag{2}$$

The origin is a persistent center if and only if it writes in one following forms:

(a) $\dot{z} = iz + Az^2 + C\bar{z}^2$ *(quadratic),*

(b) $\dot{z} = iz + Az^2 + Dz^3$ *(holomorphic),*

(c) $\dot{z} = iz + C\bar{z}^2 + G\bar{z}^3$ *(hamiltonian),*

(d) $\dot{z} = iz + Bz\bar{z} + Fz\bar{z}^2$ *(separated).*

Recall again that the general problem of obtaining all the centers for cubic differential equations is far from being solved.

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