Asymptotics of zeros of Jacobi–Sobolev orthogonal polynomials

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Abstract: *Inner products of the type $\langle f,g\rangle_S = \langle f,g\rangle_{\psi_0} + \langle f',g'\rangle_{\psi_1}$, where one of the measures ψ_0 or ψ_1 is the measure associated with the Jacobi polynomials, are usually referred to as Jacobi–Sobolev inner products. We study the behaviour of the Jacobi–Sobolev polynomials, which are orthogonal with respect to a class of Jacobi–Sobolev inner product, on the interval (-1,1). We obtain the asymptotics of the zeros of these polynomials.*

Key words: *Sobolev orthogonal polynomials, Zeros of orthogonal polynomials, Asymptotics.*

1 Introduction

Let $\{P_n^{\psi}\}_{n=0}^{\infty}$ be the sequence of monic orthogonal polynomials with respect to the inner product

$$\langle f,g\rangle_{\psi} = \int_c^d f(x)g(x)d\psi(x),$$

where $d\psi$ is a positive measure with bounded support $I=[c,d]$, $-\infty \le c < d \le \infty$. The zeros of P_n^{ψ} are simple, real and they lie inside (c,d). For more details see [8].

In this work we deal with the Sobolev inner product

$$\langle f,g\rangle_S = \int_c^d f(x)g(x)d\psi_0(x) + \int_c^d f'(x)g'(x)d\psi_1(x), \tag{1}$$

where ψ_0 and ψ_1 are positive measures with bounded support I. We denote by $\{\mathcal{S}_n\}_{n=0}^{\infty}$ the sequence of monic orthogonal polynomials with respect to the inner product (1), which are known as Sobolev orthogonal polynomials. We denote $\rho_n^{(S)} = \langle \mathcal{S}_n, \mathcal{S}_n\rangle_S$.

We denote by $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ the sequence of monic Jacobi polynomials that are orthogonal in $[-1,1]$ with respect to the measure

$$d\psi^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}dx, \quad \alpha,\beta > -1.$$

These polynomials satisfy $\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = nP_{n-1}^{(\alpha+1,\beta+1)}(x)$. We denote $\rho_n^{(\alpha,\beta)} = \langle P_n^{(\alpha,\beta)}, P_n^{(\alpha,\beta)}\rangle_{\psi^{(\alpha,\beta)}}$.

Now, we consider another family of orthogonal polynomials related to Jacobi polynomials, see [4]. Consider the measure

$$d\psi^{(\alpha,\beta,\kappa,\kappa_3)}(x) = \frac{\kappa}{\kappa-x}(1-x)^{\alpha+1}(1+x)^{\beta+1}dx + \kappa_3\delta(\kappa),$$

where $\alpha,\beta > -1$, $|\kappa| \ge 1$ and $\kappa_3 \ge 0$. We denote by $\{P_n^{(\alpha,\beta,\kappa,\kappa_3)}\}_{n=0}^{\infty}$ the sequence of monic orthogonal polynomials with respect to the inner product

$$\langle f,g\rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}} = \int_{-1}^{1} f(x)g(x)\frac{\kappa}{\kappa-x}(1-x)^{\alpha+1}(1+x)^{\beta+1}dx + \kappa_3 f(\kappa)\,g(\kappa),$$

and $\rho_n^{(\alpha,\beta,\kappa,\kappa_3)} = \langle P_n^{(\alpha,\beta,\kappa,\kappa_3)}, P_n^{(\alpha,\beta,\kappa,\kappa_3)} \rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}}$. These polynomials are related to Jacobi polynomials by

$$P_n^{(\alpha,\beta,\kappa,\kappa_3)}(x) = P_n^{(\alpha+1,\beta+1)}(x) + d_{n-1} P_{n-1}^{(\alpha+1,\beta+1)}(x), \quad n \geq 1,$$

with $P_0^{(\alpha,\beta,\kappa,\kappa_3)}(x) = 1$ and $d_{n-1} = -\dfrac{\rho_n^{(\alpha,\beta,\kappa,\kappa_3)}}{\kappa\, \rho_{n-1}^{(\alpha+1,\beta+1)}}$, see [4].

2 Jacobi–Sobolev orthogonal polynomials

We consider the Sobolev inner product introduced in [4]

$$\begin{aligned}
\langle f, g\rangle_S &= \langle f, g\rangle_{\psi^{(\alpha,\beta)}} + \kappa_1 \langle f', g'\rangle_{\psi^{(\alpha+1,\beta+1)}} + \kappa_2 \langle f', g'\rangle_{\psi^{(\alpha,\beta,\kappa,\kappa_3)}} \\
&= \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}dx + \kappa_1 \int_{-1}^{1} f'(x)g'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}dx \\
&\quad + \kappa_2 \left[\int_{-1}^{1} f'(x)g'(x)\frac{\kappa}{\kappa - x}(1-x)^{\alpha+1}(1+x)^{\beta+1}dx \;+\; \kappa_3 f'(\kappa)\, g'(\kappa) \right],
\end{aligned} \tag{2}$$

where $\alpha, \beta > -1$, $|\kappa| \geq 1$, $\kappa_2 \geq 0$, $\kappa_3 \geq 0$ and $\kappa_1 \geq -|\kappa|\kappa_2/(1+|\kappa|)$.

The sequence of monic polynomials, $\{\mathcal{S}_n\}_{n=0}^{\infty}$, orthogonal with respect to (2) is called Jacobi–Sobolev orthogonal polynomials. In [4] the authors established the relation

$$\mathcal{S}_{n+1}(x) + a_n \mathcal{S}_n(x) = P_{n+1}^{(\alpha,\beta)}(x) + b_n P_n^{(\alpha,\beta)}(x), \quad n \geq 1, \tag{3}$$

where $\mathcal{S}_0(x) = 1$, $\mathcal{S}_1(x) = P_1^{(\alpha,\beta)}(x)$, and, for $n \geq 1$,

$$b_n = \frac{n+1}{n} d_{n-1} = -\frac{n+1}{n} \frac{\rho_n^{(\alpha,\beta,\kappa,\kappa_3)}}{\kappa \rho_{n-1}^{(\alpha+1,\beta+1)}}, \qquad a_n = b_n \frac{\rho_n^{(\alpha,\beta)} + \kappa_1 n^2 \rho_{n-1}^{(\alpha+1,\beta+1)}}{\rho_n^{(S)}}. \tag{4}$$

If $\kappa_1 \neq 0$, the pair of measures $\{\psi^{(\alpha,\beta)}, \kappa_1 \psi^{(\alpha+1,\beta+1)} + \kappa_2 \psi^{(\alpha,\beta,\kappa,\kappa_3)}\}$ do not form a coherent pair according to Meijer's classification given in [5]. For $\kappa_1 = 0$ we can find some asymptotics results in [6] and [7].

Here we consider $\kappa_1 > 0$, $\alpha, \beta > -1$, $|\kappa| \geq 1$, $\kappa_2 \geq 0$ and $\kappa_3 \geq 0$.

Using similar techniques as in [3] we can show the following asymptotic behaviour for the coefficients b_n and a_n in the relation (3)

$$b = \lim_{n\to\infty} b_n = \lim_{n\to\infty} d_n = \begin{cases} -\dfrac{\Phi(\kappa)}{2}, & \text{if } \kappa_3 > 0, \\ -\dfrac{1}{2\Phi(\kappa)}, & \text{if } \kappa_3 = 0, \end{cases} \quad \text{and} \quad a = \lim_{n\to\infty} a_n = -\frac{1}{2\Phi(\tilde{\kappa})}, \tag{5}$$

where $\tilde{\kappa} = \kappa\left(1 + \frac{\kappa_2}{\kappa_1}\right)$ and Φ is the complex function defined by $\Phi(x) = x + \sqrt{x^2-1}$, for $x \in \mathbb{C} \setminus [-1,1]$. The sign of the square root in Φ is such that $|\Phi(x)| > 1$, for $x \in \mathbb{C} \setminus [-1,1]$. Also, we can show that

$$\lim_{n\to\infty} \frac{\mathcal{S}_n(x)}{P_n^{(\alpha,\beta)}(x)} = \begin{cases} \dfrac{\Phi(x) - \Phi(\kappa)}{\Phi(x) - 1/\Phi(\tilde{\kappa})}, & \text{if } \kappa_3 > 0, \\ \dfrac{\Phi(x) - 1/\Phi(\kappa)}{\Phi(x) - 1/\Phi(\tilde{\kappa})}, & \text{if } \kappa_3 = 0, \end{cases} \tag{6}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1,1]$.

3 Asymptotics of zeros of Jacobi–Sobolev orthogonal polynomials

In [2] the authors have shown that under the conditions

$$\kappa_2 \geq 2\kappa_1 \geq 0, \quad \alpha+\beta > 2 \quad \text{and} \quad \begin{cases} \alpha \leq \beta, & \text{if} \quad \kappa \leq -1, \\ \alpha \geq \beta, & \text{if} \quad \kappa \geq 1, \end{cases} \tag{7}$$

the polynomial $\mathcal{S}_n$ has n different real zeros and at least $n-1$ lie inside $(-1,1)$. Although, numerical experiments lead the authors to conjecture that this happens for all valid values of the parameters.

If we denote the zeros of $\mathcal{S}_n$ by $s_{n,i}$, $i=1,2,\ldots,n$, in decreasing order, i.e., $s_{n,n} < s_{n,n-1} < \cdots < s_{n,2} < s_{n,1}$. Then, we can have all zeros inside $(-1,1)$ or $s_{n,1} > 1$ or $s_{n,n} < -1$.

Consider $\kappa_3 > 0$ and choosing $x=\kappa$ with $|\kappa| \geq 1$ in (6), since $P_n^{(\alpha,\beta)}(\kappa) \neq 0$, we can conclude that, for $\kappa_3 > 0$,

$$\lim_{n\to\infty} s_{n,1} = \kappa, \qquad \text{if } \kappa \geq 1,$$

$$\lim_{n\to\infty} s_{n,n} = \kappa, \qquad \text{if } \kappa \leq -1.$$

The Mehler–Heine formula for monic Jacobi orthogonal polynomials is

$$\lim_{n\to\infty} \frac{2^n P_n^{(\alpha,\beta)}(\cos(x/(n+j)))}{n^{\alpha+\frac{1}{2}}} = \frac{\sqrt{\pi}}{2^\beta} x^{-\alpha} J_\alpha(x), \tag{8}$$

where $j \in \mathbb{Z}$ and J_α is the Bessel function of the first kind. It holds uniformly in every bounded region of the complex plane $\mathbb{C}$, (see [8]).

Let us define

$$R_n(x) := P_n^{(\alpha,\beta)}(x) + b_{n-1} P_{n-1}^{(\alpha,\beta)}(x) = \mathcal{S}_n(x) + a_{n-1}\mathcal{S}_{n-1}(x), \quad n \geq 1, \tag{9}$$

and $R_0(x)=1$. From (8) we can obtain a Mehler–Heine type formula for R_n.

Lemma 1 *We have for a fixed $j \in \mathbb{Z}$ and $\alpha,\beta > -1$,*

$$\lim_{n\to\infty} \frac{2^n R_n(\cos(x/(n+j)))}{n^{\alpha+\frac{1}{2}}} = (1+2b)\frac{\sqrt{\pi}}{2^\beta} x^{-\alpha} J_\alpha(x),$$

uniformly on compact subsets of the complex plane.

Proof. From (9) we get

$$\begin{aligned} \frac{2^n R_n^{(q)}\big(\cos(x/(n+j))\big)}{n^{\alpha+\frac{1}{2}}} &= \frac{2^n P_n^{(\alpha,\beta)}\big(\cos(x/(n+j))\big)}{n^{\alpha+\frac{1}{2}}} \\ &\quad + 2b_{n-1}\frac{(n-1)^{\alpha+\frac{1}{2}}}{n^{\alpha+\frac{1}{2}}} \frac{2^{n-1} P_{n-1}^{(\alpha,\beta)}\big(\cos(x/(n+j))\big)}{(n-1)^{\alpha+\frac{1}{2}}}. \end{aligned}$$

The result follows from (8) and (5). □

We need the following result that it is shown in [1].

Lemma 2 *Let $\{c_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that $\lim_{n\to\infty} c_n = c$ and $|c| < 1$. For $n \geq 0$, and $i=1,2,\ldots,n$, let $t_i^{(n)} = \prod_{j=1}^{i} c_{n-j}$ and $t_0^{(n)} = 1$. Then, there exist constants P and r where $P > 1$ and $0 < r < 1$ such that $|t_i^{(n)}| < Pr^i$ for all $n \geq 0$ and $0 \leq i \leq n$.*

Now we can obtain a Mehler–Heine type formula for the Jacobi–Sobolev orthogonal polynomials, $\mathcal{S}_n$.

Theorem 1 *We have for $\alpha, \beta > -1$,*

$$\lim_{n\to\infty} \frac{2^n \mathcal{S}_n\left(\cos(x/n)\right)}{n^{\alpha+\frac{1}{2}}} = \frac{1+2b}{1+2a}\frac{\sqrt{\pi}}{2^{\beta}} x^{-\alpha} J_{\alpha}(x), \tag{10}$$

uniformly on compact subsets of the complex plane.

Proof. From (9) we can write $\mathcal{S}_n(x) = R_n(x) - a_{n-1}R_{n-1}(x),\ n \geq 1$, then recursively we obtain

$$\mathcal{S}_n(x) = \sum_{i=0}^{n} (-1)^i u_i^{(n)} R_{n-i}(x), \quad n \geq 1, \tag{11}$$

where $u_i^{(n)} = \prod_{j=1}^{i} a_{n-j}$, for $i = 1, 2, \ldots, n$, and $u_0^{(n)} = 1$.

From (11) we can write

$$\frac{2^n \mathcal{S}_n\left(\cos(x/n)\right)}{n^{\alpha+\frac{1}{2}}} = \sum_{i=0}^{n} (-1)^i 2^i u_i^{(n)} \frac{(n-i)^{\alpha+\frac{1}{2}}}{n^{\alpha+\frac{1}{2}}} \frac{2^{n-i} R_{n-i}\big(\cos(x/n)\big)}{(n-i)^{\alpha+\frac{1}{2}}}.$$

From (5) and Lemma 2 with $t_i^{(n)} = 2^i u_i^{(n)}$ we get

$$|2^i u_i^{(n)}| = |2^i \prod_{j=1}^{i} a_{n-j}| = |2a_{n-1}\ 2a_{n-2} \cdots 2a_{n-i}| < P r^i,$$

where $0 < r < 1$ and $P > 1$.

This bound and Lemma 1 allow us to use Lebesgue's dominated convergence theorem, and we get

$$\lim_{n\to\infty} \frac{2^n \mathcal{S}_n\big(\cos(x/n)\big)}{n^{\alpha}} = (1+4b^{(q)})\sqrt{\pi}(2x)^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(x) \sum_{i=0}^{\infty} (-4a^{(q,\kappa_1,\kappa_2)})^i,$$

from where the result follows. □

According to (5) when $\kappa = 1$ we obtain $b = -1/2$. In this case the above theorem does not provide asymptotic information because the value of the limit in (10) is 0. On the other hand, when $\kappa \neq 1$ the formula (10) is useful to obtain the asymptotic behaviour of the largest zeros of these polynomials. Applying Hurwitz's theorem to (10) we get the following.

Corollary 1 *Let the parameters satisfying the conditions (7) with $\kappa \neq 1$. Let $s_{n,m} < s_{n,m-1} < \cdots < s_{n,2} < s_{n,1}$ be the m zeros of $\mathcal{S}_n$ within $(-1, 1)$. Then,*

$$\lim_{n\to\infty} n \arccos(s_{n,i}) = j_i^{(\alpha)},$$

where $0 < j_1^{(\alpha)} < j_2^{(\alpha)} < \cdots < j_m^{(\alpha)}$ denote the first m positive zeros of the Bessel function of the first kind J_{α}.

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