

On Fuzzy Negations and Automorphisms

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Abstract: *Fuzzy logic overcome the inability of classic logic to deal with vagues, uncomplete or imprecise information by admits a truth degree for propositions, typically a value in the unit interval, whereas in classic logic a proposition can be just false or true. This characteristic turn fuzzy logic adequate to model the approximate reasoning of human agents. Fuzzy logical connectives must to have the same behavior than the classical connectives in the extremes of the unit interval (boundary conditions). However it is not sufficient, also is necessary to require other properties that the classical connectives satisfies. In the particular case of fuzzy negation beyond of the boundary condition also is require that it be antitonic.*

Several studies on the properties of fuzzy negations has been produced from the seminal work of Zadeh. This paper provide some new results on the relation of the automorphism notion with the fuzzy negations. The main of this results is to provide a generalization of the notion and properties of negation-preserving automorphism.

Key-words: *Fuzzy Logic, Fuzzy negations, Automorphism, Equilibrium point.*

1 Introduction

In [31], Lotfi A. Zadeh proposed the fuzzy set theory to deal with the aspect of uncertainty found in the definition of a vagueness concept or the meaning of a term such as “young person” or “higher temperature”. Thus, in order to give a meaning of how much an expert believes that a given element belongs to a fuzzy set or equivalently satisfies a vague property, this theory considers a membership degree (a value in the unit interval $[0, 1]$). Therefore, in its subjacent logic (fuzzy logic) a proposition is not simply true or false, but has a truth degree (a value in $[0, 1]$), where 0 means that the proposition is absolutely false, and 1 that the proposition is absolutely true. Intelligent computational systems using fuzzy logics, i.e. fuzzy systems, are efficient to deal with uncertain information and therefore with approximate reasoning [6]. An important class of fuzzy systems are the fuzzy rule-based systems. The rules are described by the use of propositional connectives such as conjunction, negation and implications. But, there exists infinitely many ways to extend the classical propositional connectives to the set $[0, 1]$ such that the behavior in its extremes is as in the classical logic. Still, it is a consensus that it is not sufficient, demanding that these extensions also preserve some minimal logical properties of the classical connectives. Thus, the notions of t-norms, t-conorms, fuzzy negations, and fuzzy implications were introduced.

From the seminal work of Lotfi A. Zadeh in [31] several approaches had been proposed for fuzzy negations. Although Zadeh fuzzy negation $C(x) = 1 - x$ is the most used in fuzzy systems, there are important classes of fuzzy negation proposed with different motivations. The class of Sugeno complement is obtained from a kind of special measures defined by Michio Sugeno himself in [27], and Ronald Yager class of fuzzy negations, which results from the fuzzy unions by requiring that $N(x) \vee x = 1$ for each $x \in U$. Both, can derive most of the fuzzy negations which are used in the practice [26]. Nevertheless, other different fuzzy negations were defined in

the end of the 70's and beginning of the 80's, for example, in [18, 28, 11, 15, 25]. The axiomatic definition as it is known today for the fuzzy negation can be found in [15]. Although of fuzzy connectives has been a subject of intense research in 70's, their study is still valid nowadays. For example, for the case of fuzzy negations, see the papers [1, 5, 2, 19, 8].

The main contribution of this paper is provide a generalization of the notion of negation-preserving automorphism introduced by Mirko Navara in [22] and prove that this generalization satisfy analogous properties. Another contribution is to prove that there exists neither a lesser nor a greater strict (strong) fuzzy negation.

2 Fuzzy Negations

A function $N : U \rightarrow U$, where U denotes the unit interval $[0, 1]$, is a **fuzzy negation** if

- N1: $N(0) = 1$ and $N(1) = 0$.
- N2: If $x \leq y$ then $N(y) \leq N(x)$, $\forall x, y \in U$.

Fuzzy negations are **strict** if it satisfies the following properties

- N3: N is continuous,
- N4: If $x < y$ then $N(y) < N(x)$, $\forall x, y \in U$.

Fuzzy negations satisfying the **involution** property, i.e.

- N5: $N(N(x)) = x$, $\forall x \in U$,

are called **strong fuzzy negations**. Notice that each strong fuzzy negation is strict but the reverse is not true. For example, the fuzzy negation $N(x) = 1 - x^2$ is strict but not strong.

Notice that if N is a strong fuzzy negation, then $N = N^{-1}$.

An **equilibrium point** of a fuzzy negation N is a value $e \in U$ such that $N(e) = e$.

Remark 2.1 *Let N be a fuzzy negation. If e is an equilibrium point for N then by antitonicity of N for each $x \in U$, if $x \leq e$ then $e \leq N(x)$ and if $e \leq x$ then $N(x) \leq e$.*

Remark 2.2 *Let N be a fuzzy negation. If e is an equilibrium point for N and if $x \leq N(x)$ then $x \leq e$ and if $N(x) \leq x$ then $e \leq x$.*

George Klir and Bo Yuan in [17] proved that all fuzzy negations have at most one equilibrium point and so if a fuzzy negation N has an equilibrium point then it is unique. For example, the strict fuzzy negation $N(x) = 1 - x^2$ has $\frac{\sqrt{5}-1}{2} \cong 0.618034$ as the unique equilibrium point. However, not all fuzzy negations have an equilibrium point, for example the fuzzy negation N_{\perp} , defined below has no equilibrium point.

$$N_{\perp}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Remark 2.3 *Let N be a strict (strong) fuzzy negation. Then by continuity, N has an equilibrium point. As noted above, its equilibrium point is unique.*

Remark 2.4 *Let $e \in U$. Then there exists infinitely many fuzzy negations having e as equilibrium point. For example, if N is a strong fuzzy negation then the function $N' : U \rightarrow U$, defined as*

$$N'(x) = \begin{cases} N\left(\frac{N(e)x}{e}\right) & \text{if } x \leq e \\ \frac{N(x)e}{N(e)} & \text{if } x > e, \end{cases}$$

is a strict fuzzy negation such that $N'(e) = e$.

Analogously to a t-norm, it is also possible to establish a partial order on the fuzzy negations in a natural way, i.e. given two fuzzy negations N_1 and N_2 we say that $N_1 \leq N_2$ if for each $x \in U$, $N_1(x) \leq N_2(x)$.

Proposition 2.1 *Let N_1 and N_2 be fuzzy negations such that $N_1 \leq N_2$. Then if e_1 and e_2 are the equilibrium points of N_1 and N_2 , respectively, then $e_1 \leq e_2$.*

Proof: Let e_1 and e_2 the equilibrium points of N_1 and N_2 respectively. Suppose that $e_2 < e_1$ then $e_1 \leq N_1(e_2)$. Thus, because $N_1 \leq N_2$, $e_1 \leq N_1(e_2) \leq N_2(e_2) = e_2$ which is a contradiction. Therefore, $e_1 \leq e_2$.

Clearly, for any fuzzy negation N ,

$$N_{\perp} \leq N \leq N_{\top} \quad (1)$$

where

$$N_{\top}(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x < 1 \end{cases}$$

Notice that neither N_{\perp} nor N_{\top} are strict. Then, it is natural to ask, there exists a lesser and a greater strict (strong) fuzzy negation?

In the next subsection we will answer this question.

3 Automorphisms

A mapping $\rho : U \rightarrow U$ is an **automorphism** if it is bijective and monotonic, i.e. $x \leq y \Rightarrow \rho(x) \leq \rho(y)$ [16, 22]. An equivalent definition was given in [7], where automorphisms are continuous and strictly increasing functions $\rho : U \rightarrow U$ such that $\rho(0) = 0$ and $\rho(1) = 1$. Automorphisms are closed under composition, i.e. if ρ and ρ' are automorphisms then $\rho \circ \rho'(x) = \rho(\rho'(x))$ is also an automorphism. The inverse of an automorphism is also an automorphism. Thus, $(Aut(U), \circ)$, where $Aut(U)$ is the set of all automorphisms, is a group, with the identity function being the neutral element and ρ^{-1} being the inverse of ρ [14].

Let ρ be an automorphism and N be a fuzzy negation. The **action of ρ on N** , denoted by N^{ρ} , is defined as follows

$$N^{\rho}(x) = \rho^{-1}(N(\rho(x))) \quad (2)$$

Proposition 3.1 *Let $N : U \rightarrow U$ be a fuzzy negation. If e is an equilibrium point of N then $\rho^{-1}(e)$ is an equilibrium point of N^{ρ} .*

Proof: Trivially, $N^{\rho}(\rho^{-1}(e)) = \rho^{-1}(N(\rho(\rho^{-1}(e)))) = \rho^{-1}(N(e)) = \rho^{-1}(e)$.

Proposition 3.2 *Let $N : U \rightarrow U$ be a fuzzy negation and $\rho : U \rightarrow U$ be an automorphism. Then N^{ρ} is also a fuzzy negation. Moreover, if N is strict (strong) then N^{ρ} is also strict (strong).*

Proof: Let $x, y \in U$.

- N1: Trivially, $N^{\rho}(0) = \rho^{-1}(N(\rho(0))) = \rho^{-1}(N(0)) = 1$ and $N^{\rho}(1) = \rho^{-1}(N(\rho(1))) = \rho^{-1}(N(1)) = 0$.
- N2: If $x \leq y$ then $\rho(x) \leq \rho(y)$. Thus, by N2, $N(\rho(y)) \leq N(\rho(x))$ and so $\rho^{-1}(N(\rho(y))) \leq \rho^{-1}(N(\rho(x)))$. Therefore, $N^{\rho}(y) \leq N^{\rho}(x)$.
- N3: Composition of continuous functions is also continuous.
- N4: Analogous to N2.

- $N5: N^\rho(N^\rho(x)) = \rho^{-1}(N(\rho^{-1}(\rho(N(\rho(x))))) = \rho^{-1}(N(N(\rho(x)))) = \rho^{-1}(\rho(x)) = x.$

Proposition 3.3 *Let N be a strict (strong) fuzzy negation and the automorphism $\rho(x) = x^2$. Then, $N < N^\rho$ and $N^{\rho^{-1}} < N$.*

Proof: Note that $\rho^{-1}(x) = \sqrt{x}$. Since $x^2 < x$ for each $x \in (0, 1)$, then $N(x) < N(x^2)$ and so $\rho^{-1}(N(x)) < \rho^{-1}(N(\rho(x))) = N^\rho(x)$. But, once that $x < \sqrt{x}$ for each $x \in (0, 1)$, we have that $N(x) < N^\rho(x)$ for each $x \in (0, 1)$. So, $N < N^\rho$. The proof that $N^{\rho^{-1}} < N$ is analogous.

Corollary 3.1 *There exists neither a lesser nor a greater strict (strong) fuzzy negation.*

Proof: Straightforward from propositions 3.2 and 3.3.

The following proposition stated by Enric Trillas in [28], presents a strong relation between automorphism and strong fuzzy negations.

Proposition 3.4 *A function $N : U \rightarrow U$ is a strong fuzzy negation if and only if there exists an automorphism ρ such that $N = C^\rho$, where C is the strong fuzzy negation $C(x) = 1 - x$.*

Proof: See [28].

This proposition was generalized by János Fodor in [12] for strict fuzzy negations.

Proposition 3.5 *A function $N : U \rightarrow U$ is a strict fuzzy negation if and only if there exist automorphisms ρ_1 and ρ_2 such that $N = \rho_1 \circ C \circ \rho_2$, where C is the strong fuzzy negation $C(x) = 1 - x$.*

Proof: See [12].

A possible application of the Corollary 3.1 could be the establishment of an alternative characterization to propositions 3.4 and 3.5 for strong and strict fuzzy negations, respectively.

4 N -Preserving automorphism

Mirko Navara, in order to answer a question stated by himself in [21], introduced in [22] the notion of **negation-preserving automorphism** as being an automorphism which comutes with the usual negation $C(x) = 1 - x$, i.e. an automorphism ρ such that $\rho(C(x)) = C(\rho(x))$. From then, several applications for this concept has been made. For example, in [3] this notion was used to provide a characterization of strict Frank triangular norms and T-measures and in [23, Theorem 13.9.4] was used for characterization of charges on general tribes. Here it is introduced a natural generalization of this notion which could be applied, for example, to characterize a family of strict Frank triangular norms, T-measures and charges of tribes.

Let N be a fuzzy negation. An automorphism ρ is **N -preserving automorphism** if for each $x \in U$,

$$\rho(N(x)) = N(\rho(x)). \quad (3)$$

The next proposition is a generalization of [22, Proposition 4.2].

Proposition 4.1 *Let N be a strong fuzzy negation and ρ be an automorphism on $[0, e]$, i.e. a continuous increasing function such that $\rho(0) = 0$ and $\rho(e) = e$, where e is the unique equilibrium point of N . Then $\rho^N : U \rightarrow U$, defined by*

$$\rho^N(x) = \begin{cases} \rho(x) & \text{if } x \leq e \\ N(\rho(N(x))) & \text{if } x > e, \end{cases} \quad (4)$$

is an N -preserving automorphism. All N -preserving automorphisms are of this form.

Proof: If $x < e$ then by $N4$, $e = N(e) < N(x)$ and so

$$\begin{aligned}\rho^N(N(x)) &= N(\rho(N(N(x)))) && \text{because } N(x) > e \\ &= N(\rho(x)) && \text{because } N \text{ is strong} \\ &= N(\rho^N(x)) && \text{because } x \leq e.\end{aligned}$$

If $x > e$ then by $N4$, $N(x) < e$ and so

$$\begin{aligned}\rho^N(N(x)) &= \rho(N(x)) && \text{because } N(x) < e \\ &= N(N(\rho(N(x)))) && \text{because } N \text{ is strong} \\ &= N(\rho^N(x)) && \text{because } x > e.\end{aligned}$$

If $x = e$ then, trivially, $\rho^N(N(x)) = e = N(\rho^N(x))$.

On the other hand, if ρ' is an N -preserving automorphism then $\rho : [0, e] \rightarrow [0, e]$ defined by $\rho(x) = \rho'(x)$ is such that $\rho(e) = \rho'(N(e)) = N(\rho'(e)) = N(\rho(e))$ and so $\rho(e) = e$, the other properties that show that ρ is an automorphism on $[0, e]$ are inherited from ρ' which is an automorphism. Thus, if $x \leq e$ then $\rho'(x) = \rho^N(x)$. If $x > e$ then

$$\begin{aligned}\rho'(x) &= \rho'(N(N(x))) && \text{because } N \text{ is strong} \\ &= N(\rho'(N(x))) && \text{by equation (3)} \\ &= \rho^N(x) && \text{by equation (4)}\end{aligned}$$

Therefore, $\rho' = \rho^N$, i.e. all N -preserving automorphism have the form of equation (4).

Proposition 4.2 *Let N be a strong fuzzy negation and ρ be an automorphism on $[0, e]$, where e is the equilibrium point of N . Then $\rho^{N^{-1}}$ is an N -preserving automorphism.*

Proof: By Proposition 4.1 ρ^N is an N -preserving automorphism. Let $x \in U$.

$$\begin{aligned}\rho^{N^{-1}}(N(x)) &= \rho^{N^{-1}}(N(\rho^N(\rho^{N^{-1}}(x)))) \\ &= \rho^{N^{-1}}(\rho^N(N(\rho^{N^{-1}}(x)))) && \text{by equation (3)} \\ &= N(\rho^{N^{-1}}(x))\end{aligned}$$

So by equation (3), $\rho^{N^{-1}}$ is also an N -preserving automorphism.

5 Final Remark

Is indisputable the importance of the negation in logic. Several papers and some books (e.g. [30, 13, 20, 24]) and workshops (e.g. the Workshop on “Negation in Constructive Logic”, Dresden, July 2004; Workshop on “Negation and Denial”, Lisbon, July 2008; and Workshop on “Aspectos Lógicos da Negação”, Natal, April 2009) has been devoted to analyse the negation in a determine logic. In fuzzy logics the negation, depending how it is defined, can to incorporate the main characteristic of the negation of other logics, such as the classic [4], intuitionistic [2] and paraconsistent [9]. More over, fuzzy negation play an important role in the broad as well as narrow sense. In the broad sense, because negation is very used in fuzzy rule-based systems, and in the narrow sense, because it satisfy several interesting properties and is used, for example, to define a t-conorm (the connective fuzzy which model disjunction) from a t-norm (the connective fuzzy which model a conjunction). In fact, several studies has been made associating fuzzy negation or classes of them, with other notions (see for example, [10, 29, 7, 26, 19]).

This paper presents an study on fuzzy negations and their relation with the notion of automorphism. We proved that the action of an automorphism on fuzzy negation preserve the involutive and strictness properties. This result was used for us to prove that there not exists a lesser and a greater strict and strong fuzzy negation. We also extend the notion of negation-preserving automorphism introduced by Mirko Navara in [22] by consider automorphism which

computes with the arbitrary fuzzy negation instead of just with the Zadeh complement. We generalize the main results of Mirko Navara in [22] providing in this way a characterization of N -preserving automorphism. With it we hope have contribute for a better knowledge of the properties of fuzzy negations as well as to show how much interesting and troublemaker is this operator.

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