

# A Spatial Model to Describe Foot and Mouth Disease Dissemination\*

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## Abstract

In this work we propose a spatial model to analyze the foot and mouth dissemination in Brazil. The model aims to study this dissemination based on a system of partial differential reaction-diffusion equations taking into account susceptible, infected (clinical and subclinical) and removed animal subpopulations. Diffusion and advection are allowed for susceptible, infected subclinical and removed subpopulation. The traveling wave solutions of the model are studied to determine the speed of the disease dissemination. This wave speed is obtained as a function of the model's parameters, in order to assess the control strategies.

**Key-words:** Foot and mouth disease – Reaction-Diffusion Equation – Traveling waves – Wave speed

## 1 Model for Foot and Mouth disease (FMD)

We present a spatial model for FMD propagation and the analysis of the corresponding spatially homogeneous model for FMD disease.

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## 1.1 Model for the spatial FMD dissemination

We propose a spatially homogeneous model for the animal population, the density of which is denoted by  $N(t)$ . The population was divided into susceptible, infective (clinical and subclinical) and recovered subpopulations,  $S(t)$ ,  $I_s(t)$ ,  $I_c(t)$  and  $R(t)$ , respectively. The total population is  $N(t) = S(t) + I_s(t) + I_c(t) + R(t)$ .

The total population density is allowed to vary,  $\phi$  is the constant recruitment rate due to birth and migration (movements of animals between farms), and the death rate is  $\mu$ . The differential equation for the population irrespective of FMD is, then,

$$\frac{dN}{dt} = \phi - \mu N.$$

The infection rate per susceptible individual is proportional to the infected subpopulation, and is given by:  $\beta(I_s + I_c)$ . It is assumed that subclinical infected class pass through the clinical infected class at a rate  $\gamma$ . The clinical infected class is recovered at a rate  $\epsilon$ . A treatment in subclinical infected subpopulation is considered, and they pass through recovered class at a rate  $\tau$ . A death rate due to the sacrifice of clinical infected class is considered denoted by  $\delta$ . Vaccination treatment and loss of immunity are considered and denoted by  $\nu$  and  $\Pi$ .

From now on we consider the spatio-temporal dependence on the populations, e.g.  $N(x, t)$  and the respective subpopulations. The diffusion in susceptible and recovered classes is denoted by  $D_1$ , and  $D_2$  is designed for the diffusion of the subclinical infected class. The clinical infected subpopulation is considered sessile, due to vigilance control. The advection coefficients are denoted by  $\lambda_1$  for the susceptible and recovered classes and  $\lambda_2$  for the subclinical infected class.

Based on the above assumptions and definitions of the parameters, the spatial model is the following:

$$\frac{\partial S}{\partial t} = D_1 \frac{\partial^2 S}{\partial x^2} - \lambda_1 \frac{\partial S}{\partial x} + \phi - \beta S(I_s + I_c) - \mu S - \nu S + \Pi R \quad (1)$$

$$\frac{\partial I_s}{\partial t} = D_2 \frac{\partial^2 I_s}{\partial x^2} - \lambda_2 \frac{\partial I_s}{\partial x} + \beta S(I_s + I_c) - \mu I_s - \gamma I_s - \tau I_s \quad (2)$$

$$\frac{\partial I_c}{\partial t} = \gamma I_s - (\mu + \epsilon + \delta) I_c \quad (3)$$

$$\frac{\partial R}{\partial t} = D_1 \frac{\partial^2 R}{\partial x^2} - \lambda_1 \frac{\partial R}{\partial x} + \epsilon I_c - \mu R + \nu S + \tau I_s - \Pi R. \quad (4)$$

## 1.2 Model for the spatially homogeneous FMD dynamics

The spatially homogeneous model corresponding to the system (1) - (4) is the following:

$$\frac{dS}{dt} = \phi - \beta S(I_s + I_c) - \mu S - \nu S + \Pi R \quad (5)$$

$$\frac{dI_s}{dt} = \beta S(I_s + I_c) - \mu I_s - \gamma I_s - \tau I_s \quad (6)$$

$$\frac{dI_c}{dt} = \gamma I_s - (\mu + \epsilon + \delta) I_c \quad (7)$$

$$\frac{dR}{dt} = \epsilon I_c - \mu R + \nu S + \tau I_s - \Pi R. \quad (8)$$

The system of equations (5) - (8) has two steady states. The first is the disease-free equilibrium point given by:

$$P_0 = (\hat{S}, 0, 0, \hat{R}),$$

with  $\hat{S}$  and  $\hat{R}$  given by:

$$\hat{S} = \frac{\phi(\mu + \Pi)}{(\mu + \nu)(\mu + \Pi) - \Pi\nu}, \quad \hat{R} = \frac{\nu\phi}{(\mu + \nu)(\mu + \Pi) - \Pi\nu},$$

where  $\hat{S}$  and  $\hat{R}$  are always greater than zero.

The second one is the endemic state:

$$P_1 = (S^*, I_s^*, I_c^*, R^*),$$

where  $S^*$ ,  $I_s^*$ ,  $I_c^*$  and  $R^*$  are given by:

$$S^* = \frac{(\delta + \epsilon + \mu)(\gamma + \mu + \tau)}{\beta(\delta + \epsilon + \gamma + \mu)}, \quad I_s^* = \frac{\phi \left[ 1 - \frac{1}{R_0} \right]}{[\mu + \gamma + \tau] - \frac{\Pi}{\mu + \Pi} \left[ \frac{\epsilon\gamma}{\mu + \epsilon + \gamma} + \tau \right]},$$

$$I_c^* = \frac{\gamma I_s^*}{\mu + \epsilon + \delta}, \quad R^* = \frac{\left[ \tau + \frac{\epsilon\gamma}{\mu + \epsilon + \delta} \right] I_s^* + \nu S^*}{\mu + \Pi}.$$

Since  $[\mu + \gamma + \tau] - \frac{\Pi}{\mu + \Pi} \left[ \frac{\epsilon\gamma}{\mu + \epsilon + \gamma} + \tau \right]$  is always positive, hence a positive solution always exists for  $R_0 > 1$ , where

$$R_0 = \frac{\beta(\delta + \epsilon + \gamma + \mu)}{(\delta + \epsilon + \mu)(\gamma + \mu + \tau)} \frac{\phi(\mu + \Pi)}{(\mu + \nu)(\mu + \Pi) - \Pi\nu} \quad (9)$$

is the Basic Reproductive Number. For  $R_0 < 1$  the disease free equilibrium point  $P_0$  is locally asymptotically stable, otherwise the endemic state  $P_1$  is stable.

The following Theorem, with regard to two equilibrium points, is established:

**Theorem 1.1** *The disease free equilibrium  $P_0$  is unique and locally asymptotically stable for  $R_0 < 1$ . When  $R_0 > 1$ ,  $P_0$  becomes unstable, and a new endemic equilibrium  $P_1$  appears, which is locally asymptotically stable.*

## 2 Traveling waves solution

In this section we study the geographic propagation of FMD using the same method applied to describe the dissemination of rabies among foxes [1], [2], that is, we determine the minimum wave speed connecting the disease free equilibrium point to the endemic state. The solution corresponding to the minimum wave speed of the system of equations (1)-(4) describes the observed biological waves, see Volpert and Sandstede [4], [5].

The traveling waves solution, when it exists, can be set in the usual form [3]:

$$(s(x, t), i_s(x, t), i_c(x, t), r(x, t)) = (s(z), i_s(z), i_c(z), r(z)),$$

where  $z = x + ct$ . In this new variable, the equations (1)-(4) are transformed into:

$$c \frac{ds}{dz} = D_1 \frac{d^2s}{dz^2} - \lambda_1 \frac{ds}{dz} + \phi - \beta s(i_s + i_c) - \mu s - \nu s + \Pi r \quad (10)$$

$$c \frac{di_s}{dz} = D_2 \frac{d^2i_s}{dz^2} - \lambda_2 \frac{di_s}{dz} + \beta s(i_s + i_c) - \mu i_s - \gamma i_s - \tau i_s \quad (11)$$

$$c \frac{di_c}{dz} = \gamma i_s - (\mu + \epsilon + \delta) i_c \quad (12)$$

$$c \frac{dr}{dz} = D_1 \frac{d^2r}{dz^2} - \lambda_1 \frac{dr}{dz} + \epsilon i_c - \mu r + \nu s + \tau i_s - \Pi r. \quad (13)$$

Defining the variables  $u_1 = \frac{ds}{dz}$ ,  $u_2 = \frac{di_s}{dz}$ , and  $u_3 = \frac{dr}{dz}$ , the corresponding first order ordinary differential equations with respect to variable  $z$  of the system (10)-(13) is:

$$\frac{ds}{dz} = u_1, \quad (14)$$

$$\frac{du_1}{dz} = \frac{1}{D_1} [(c + \lambda_1)u_1 - \phi + \beta s(i_s + i_c) + \mu s + \nu s - \Pi r] \quad (15)$$

$$\frac{di_s}{dz} = u_2, \quad (16)$$

$$\frac{du_2}{dz} = \frac{1}{D_2} [(c + \lambda_2)u_2 - \beta s(i_s + i_c) + \mu i_s + \gamma i_s + \tau i_s] \quad (17)$$

$$\frac{di_c}{dz} = \frac{1}{c} [\gamma i_s - (\mu + \epsilon + \delta) i_c] \quad (18)$$

$$\frac{dr}{dz} = u_3 \quad (19)$$

$$\frac{du_3}{dz} = \frac{1}{D_1} [(c + \lambda_1) u_3 - \epsilon i_c + \mu r - \nu s - \tau i_s + \Pi r], \quad (20)$$

where the boundary conditions are:

$$\lim_{z \rightarrow -\infty} (s_a(z), u_1(z), i_a(z), u_2(z), i_v(z), n_a(z), u_3(z)) = (\hat{S}, 0, 0, 0, 0, \hat{R}, 0) \quad (21)$$

and

$$\lim_{z \rightarrow \infty} (s_a(z), u_1(z), i_a(z), u_2(z), i_v(z), n_a(z), u_3(z)) = (S^*, 0, I_s^*, 0, I_c^*, R^*, 0). \quad (22)$$

The zeros in both equilibrium points merits some comments. The three zeros in the second equilibrium point correspond to derivatives of the subpopulations  $s$ ,  $i_s$  and  $r$ . However, the first equilibrium point has two more zeros corresponding to infectious populations regarding clinical and subclinical cases, which must not assume negative values. Due to this constraint, we impose to the linear system solutions that must not oscillate, i.e., the eigenvalues corresponding to this equilibrium point must assume real values.

The roots of the characteristic polynomial regarding to the linear system at the equilibrium point  $(s_a, u_1, i_a, u_2, i_v, n_a, u_3) = (\hat{S}, 0, 0, 0, 0, \hat{R}, 0)$  are the roots of the polynomials  $Q_1(\lambda)$ ,  $Q_2(\lambda)$  and  $P(\lambda)$ , where:

$$Q_1(\lambda) = D_1 \lambda^2 - (c + \lambda_1) \lambda - \mu, \quad (23)$$

$$Q_2(\lambda) = D_1 \lambda^2 - (c + \lambda_1) \lambda - (\mu + \nu + \Pi) \quad (24)$$

and

$$P(\lambda) = D_2 \lambda^3 + A \lambda^2 + B \lambda + C, \quad (25)$$

where the coefficients are

$$A = -(c + \lambda_2) + \frac{(\delta + \epsilon + \mu) D_2}{c}$$

$$B = -(\tau + \gamma + \mu) + \beta \hat{S} - \frac{(\delta + \epsilon + \mu)(c + \lambda_2)}{c}$$

$$C = \frac{(\delta + \epsilon + \mu)(\gamma + \mu + \tau)}{c} (R_0 - 1),$$

with  $R_0$  being given by (9). Polynomials  $Q_1(\lambda)$  and  $Q_2(\lambda)$  always have real roots. Then the polynomial  $P(\lambda)$  must determine the conditions for the existence of the minimum speed, that is, the eigenvalues are real. The minimum velocity is determined by the condition that the

polynomial evaluated at the unique local minimum,  $\lambda_+$ , must be zero, that is,  $P(\lambda_+) = 0$ , where:

$$\lambda_+ = \frac{1}{3D_2} \left\{ -A + \sqrt{A^2 - 3B} \right\}.$$

### 3 Conclusion

In this paper we developed a spatial propagation model to understand the dissemination of the FMD. For the spatially homogeneous dynamics we determined, in non dimensional parameters, the threshold value:

$$R_0 = \frac{\beta(\delta + \epsilon + \gamma + \mu)}{(\delta + \epsilon + \mu)(\gamma + \mu + \tau)} \frac{\phi(\mu + \Pi)}{(\mu + \nu)(\mu + \Pi) - \Pi\nu}. \quad (26)$$

When  $R_0$  is greater than one, the endemic state of the disease exists. We study the wave speed for the traveling waves connecting this endemic point with the disease free equilibrium point. An equation for the minimum speed was determined as a function of the parameters of the model and the threshold  $R_0$ .

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