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Variational Approach for the Analytic Solution of Miscible Displacement Model - Linear Case

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Abstract: *In this paper, we show the existence of a solution for a system of partial differential equations, a second order elliptic equation for the pressure and a hyperbolic-dominated transport-diffusion equation for the concentration of one of the fluids. As an argument to prove the existence and uniqueness of the triple; pressure, velocity and concentration, the theory of monotone operators and tools from Variational Analysis are used. We also obtain the uniqueness of the ordered pair; velocity and concentration, independent of the value of the viscosity.*

Key words: *Variational Analysis, Maximal and Monotone Operators, Miscible Displacement, Porous Media.*

1 Introduction

Many physical problems are modeled by the transport-dominated diffusive systems such as the simulation of multiphase flows in porous media. It constitutes a multidisciplinary field of research, in which Physics makes its contribution building models. These models, like a tertiary recovery in petroleum reservoirs, in general includes a system of partial differential equations (PDE), representing principles of conservation of mass, momentum or energy. Similar models can be found in the works of Angermann and Knabner in [1], as well as Ewing and Wang in [6].

The theoretical analysis of the system presented many difficulties. One of the biggest issues was the fact that the elliptic and hyperbolic-parabolic equations were coupled. Therefore, the first theoretical attempt to study the existence of the coupled system, was to find a solution of the uncoupled system, in the sense that we considered the term of viscosity corresponding to the elliptic equation as constant. Despite this simplification, the result presented features that implied the use of different approaches to obtain the solution in the part of hyperbolic-parabolic equation. Discovering the linear functional such that its Fréchet subdifferential is the operator associated with this equation, was one of biggest challenges. In addition, this operator did not fit the hypothesis of the classical results for operators of monotone type and, consequently, the techniques of Variational Analysis of non-convex type appeared as the best tools to use.

We take a tour of the process to reach this initial result of our research, starting with the origin of the mathematical model (Section 2), and ending with the theoretical approach and main theorem (Section 3). Finally, in Section 4 we prove the main result.

2 The miscible displacement model

Let the oil reservoir be represented by a bounded domain $\Omega \subset \mathbb{R}^2$. Then, in a gravity-free environment, the governing equations of a standard model for miscible displacement of a binary

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mixture constitute a nonlinear coupled system of partial differential equations as follows:

$$\operatorname{div}(u) = q, \qquad u = \frac{-\mathcal{K}(\mathbf{x})}{\mu(c)}\nabla p, \tag{1}$$

$$\phi\frac{\partial c}{\partial t} + \operatorname{div}(uc - D(u)\nabla c) = \widetilde{c}q. \tag{2}$$

The first equation (1) represents the mass conservation for the total fluid (oil plus solvent), often called the pressure equation. The Darcy's velocity, $u = (u_1, u_2)$, gives a volumetric convective flow rate that is proportional to the pressure gradient, ∇p, and inversely proportional to the viscosity of the fluid, $\mu = \mu(c)$. Notice that the coefficients of the elliptic equation depend on the solvent concentration, c. The variable $\mathcal{K}(\mathbf{x})$ is the rock permeability, $\mathbf{x} = (x, y) \in \Omega$, which quantifies the capacity of the rock to transmit fluid, and q is the total volumetric flow rate at the well.

The second one (2), called the concentration equation, represents the mass conservation for the solvent component. Their coefficients depend on the pressure through the Darcy velocity. Here, the porosity of the medium is denoted by ϕ, which is a positive constant; $\widetilde{c}$ is the specified concentration at an injection well and the resident concentration at a producer; meanwhile D is the diffusion-dispersion tensor given by:

$$\text{for } |u| \neq 0,\ D = D(u) = \phi d_m I + \frac{d_\ell}{|u|}\begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix} + \frac{d_t}{|u|}\begin{pmatrix} u_2^2 & -u_1u_2 \\ -u_1u_2 & u_1^2 \end{pmatrix}, \tag{3}$$

and ϕdmI otherwise. Here $d_\ell \gg d_t \geq 0$ are longitudinal and transverse dispersion coefficients, respectively, and $d_m > 0$ is the molecular diffusion coefficient. Notice that taking $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ a nonzero vector, we have

$$(x_1, x_2)D(u)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1^2 + x_2^2)(\phi d_m + d_t|u|) + \frac{(dl - dt)}{|u|}(u_1x_1 + u_2x_2)^2.$$

The last expression is strictly positive since $d_m > 0$.

This system needs initial and boundary conditions. We suppose the periphery of the reservoir to be impermeable, assuming "no flow" boundary conditions, together with a Dirichlet boundary condition at the external boundary Γ_D of the production well. If we denote by Γ_N, the other part of the boundary of Ω, and T is a positive number, then the boundary conditions are summarized as follows:

$$\begin{cases} u \cdot n = 0, \ \mathbf{x} \in \Gamma_N, \\ p = 0, \ \mathbf{x} \in \Gamma_D, \end{cases} \tag{4}$$

$$(uc - D(u)\nabla c) \cdot n = 0, \ \mathbf{x} \in \partial\Omega, t \in J = [0, T], \tag{5}$$

where n is the unit outward normal vector to boundary of Ω.

The incompressibility of the fluids requires the following condition

$$\int_\Omega q\, d\mathbf{x} = \int_\Omega \operatorname{div}(u)\, d\mathbf{x} = \int_{\partial\Omega} u \cdot n\, ds = 0. \tag{6}$$

The initial condition is given by

$$c(\mathbf{x}, 0) = c_0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{7}$$

(1) From all these considerations, and the conclusions (2), (4), (5), (7), we shall study existence and uniqueness of the following system of equations:

$$\begin{cases} u = \frac{-\mathcal{K}(\mathbf{x})}{\mu}\nabla p; \ \ \operatorname{div}(u) = q \\ u \cdot n = 0; \ p = 0 \text{ on } \Gamma_D \times J, \\ \phi\frac{\partial c}{\partial t} + \operatorname{div}(uc - D(u)\nabla c) = \widetilde{c}q, \ (\mathbf{x}, t) \in \Omega \times J; \\ c(\mathbf{x}, 0) = c_0(\mathbf{x}), \ \mathbf{x} \in \Omega \\ (uc - D(u)\nabla c) \cdot n = 0, \ \mathbf{x} \in \partial\Omega, t \in J \end{cases} \tag{L}$$

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3 Theoretical approach and main result

For the theoretical analysis of the uncoupled system (L), we will assume the following hypotheses:

(h_1) The domain $\Omega \subset \mathbb{R}^2$ is bounded and there are Γ_D and Γ_N, subsets of positive one dimensional measure, such that

$$\Gamma_D \cup \Gamma_N = \partial\Omega \text{ and } \Gamma_D \cap \Gamma_N = \emptyset.$$

Besides, there exist Ω_I and Ω_P, subsets of Ω with positive measure, which represent the injection and the producer well of our model, respectively.

Remark 3.1 *Following the model we describe in section 2, we will consider Ω as a rectangle in $\mathbb{R}^2$. However, from the theoretical point of view, those features are not necessary. Therefore, and considering the mixed boundary conditions, a sketch of our model of Ω will be as shown by Figure 2.*

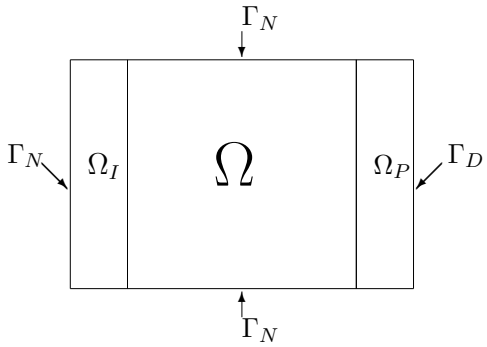

Figure 2: sketch of our model of Ω.

(h_2) The initial data c_0 verifies: $c_0 \in H^1(\Omega)$ and $0 \leq c_0(\mathbf{x}) \leq 1$, a.e. $\mathbf{x}$ in Ω. Besides, if (p, u) is a solution of system (1) with boundary conditions (4), c_0 verifies the boundary condition $D(u)\nabla c_0 \cdot n = 0,\ \mathbf{x} \in \partial\Omega$.

(h_3) The function $q \in L^2(\Omega)$ verifies (6). Besides, it is positive in Ω_I, negative Ω_P, and 0 otherwise.

(h_4) The function $\mathcal{K} > 0$ is a piecewise constant.

(h_5) $\mu(c) = \mu \in \mathbb{R}$.

There are many reasons why we cannot expect to find classical solutions of the system (L): The regularity of the domain (that we expect having corners), the mixed boundary conditions, and the term $\mathcal{K}$, which we will assume to be in the set $L^\infty(\Omega)$. As a consequence, we introduce the following notion of solution.

Definition 3.1 *We say that (p, u) is a weak solution of (1) with the boundary conditions (4) if the pair verifies:*

- $u \in H(\mathbf{div}, \Omega)$, *where* $H(\mathbf{div}, \Omega) = \{\xi \in \left[L^2(\Omega)\right]^2,\ \mathrm{div}(\xi) \in L^2(\Omega),\ \xi \cdot n = 0 \text{ on } \Gamma_N\}$.
- $p \in V = \{\varphi \in H^1(\Omega), \varphi = 0 \text{ on } \Gamma_D\}$.
- *For all test function $\xi \in V$ we have*

$$\int_\Omega \frac{\mathcal{K}(\mathbf{x})}{\mu}\ \nabla p\ \nabla\xi d\mathbf{x} - \int_\Omega q \cdot \xi d\mathbf{x} = 0.$$

- *The triple (p, u, c) is called a solution of the system (L) if (p, u) is a weak solution of (1) with the boundary conditions (4), $c \in H_1(\Omega)$, $\frac{dc}{dt} \in (J; H^{-1}(\Omega))$, and*

$$\langle \frac{dc}{dt}, v\rangle + \int_\Omega D(u)\nabla c \cdot \nabla v \; d\mathbf{x} - \int_\Omega c \cdot u \nabla v - \int \tilde{c} v q \; d\mathbf{x} = 0, \forall v \in H_1(\Omega),$$

where $H^{-1}(\Omega)$ is the dual of $H^1(\Omega)$, and $\langle \cdot, \cdot \rangle$ represents the duality pairing.

Our main result is the following:

Theorem 3.1 *Assume the hypotheses (h_1) - (h_5). Then, the system (L) has a unique weak solution (p, u, c), where $c(t)$ is a Lipschitz continuous function for all $t \in J$.*

Furthermore, denoting by (L_μ) and $(L_{\hat{\mu}})$ the system (L) with constant viscosity μ and $\hat{\mu}$, respectively; (p, u, c) and $(\hat{p}, \hat{u}, \hat{c})$ the respective solution. Then, we have $(u, c) = (\hat{u}, \hat{c})$.

Completing the tour that we started in Section 2, we point out that nowadays we are working on the conclusion of an existence result of the "maximal viscosity" solution for the nonlinear coupled system. In this case, we consider $\mu = \mu(c(t, \mathbf{x}))$, in the system (L), with initial viscosity given by $\mu(c_0(\mathbf{x}))$.

4 Proof of Theorem 3.1

Through this section, we denote $\int_\omega f(\mathbf{x}) \; d\mathbf{x} = \int_\omega f$, where $\omega \subset \Omega$.

We start with the proof of two claims:

Claim 4.1 *Let u be the solution for the elliptic boundary problem (1), (4) and (5). Let A be the operator $A : Dom(A) \to H^1(\Omega)$ defined by*

$$\langle A(c), v\rangle = -\int_\Omega (uc - D(u)\nabla c) \cdot \nabla v - \int_\Omega \tilde{c} v q, \forall v \in H^1(\Omega), \tag{8}$$

where $Dom(A) = \{c \in H^1(\Omega), \text{ verifying } D(u)\nabla c \cdot n = 0, \; \mathbf{x} \in \partial\Omega \text{ and } 0 \le c \le 1 \text{ in } \Omega \times J\}$.

Then, the operator A is monotone.

Proof: Let c_1 and $c_2 \in H^1(\Omega)$. It is enough to show that the operator

$$\langle R(c), v\rangle = -\int_\Omega uc \cdot \nabla v - \int_\Omega \tilde{c} q v, \; v \in H^1(\Omega), \tag{9}$$

is monotone, since $D(u)$ is symmetric and defined positive.

$$\begin{array}{rcl} \langle R(c_1) - R(c_2), c_1 - c_2\rangle & = & -\int_\Omega (c_2 - c_1)u \cdot \nabla(c_2 - c_1) + \int_\Omega (\tilde{c_2} - \tilde{c_1})(c_1 - c_2)q \\ & = & I_1 + I_2. \end{array} \tag{10}$$

For the first integral I_1 in (10), we use the field F defined by $F = c^2 u = (c^2 u_1, c^2 u_2)$ we notice that

$$\text{div}(F) = 2c\nabla c \cdot u + c^2 \; \text{div}(u),$$

remarking that the last equality has sense, since $c \in H^2(\Omega)$ and $u \in H(\text{div}, \Omega)$. As a consequence of these assertions and according to the boundary conditions (4), we obtain that

$$\int_\Omega (c_1 - c_2)u \cdot \nabla(c_1 - c_2)) = -\left(\frac{1}{2}\int_{\Omega_I} (c_1 - c_2)^2 q + \frac{1}{2}\int_{\Omega_P} (c_1 - c_2)^2 q\right). \tag{11}$$

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Working now with the second integral I_2 of (10), we are going to use the definition of the function $\tilde{c}$, described in Section 2, to obtain

$$\begin{aligned}\int_\Omega(\tilde{c_2}q-\tilde{c_1}q)(c_1-c_2) &= \int_{\Omega_I}(q-q)(c_1-c_2)+\int_{\Omega_P}q(c_2-c_1)(c_1-c_2)\\ &= -\int_{\Omega_P}q(c_1-c_2)^2.\end{aligned} \tag{12}$$

Putting all the conclusions together (10), (11) and (12), and using hypothesis (h_2) that q is negative in Ω_p, one has

$$\langle L(c_1)-L(c_2),c_1-c_2\rangle = -\frac{1}{2}\int_{\Omega_P}(c_1-c_2)^2q+\frac{1}{2}\int_{\Omega_I}(c_1-c_2)^2q\geq 0$$

which proves that the operator is monotone. ■

Claim 4.2 *In the same conditions of Claim 4.1, the operator A is maximal monotone.*

Proof: Our goal is to show that the operator A is the sum of two operators, one Lipschitz on the whole space $H^1(\Omega)$ and the other of the type $\partial\Theta$, where $\partial\Theta$ is the Fréchet subdifferencial of a functional $\Theta : H^1(\Omega)\to\mathbb{R}\cup\{\infty\}$, lower semicontinuous (lsc) and proper, i.e. $Dom(\Phi)\neq\varnothing$. Then, we will have A maximal monotone because of [3, Lm 2.4].

Indeed, the operator (9) is Lipschitz over $H^1(\Omega)$. On the other hand, defining the functional

$$\Theta(c)=\int_\Omega D(u)\nabla c\cdot\nabla c, c\in Dom(A),$$

it is Fréchet differentiable. In fact, for $v\in H^1(\Omega)$ such that the function $c+v\in Dom(A)$, we have

$$\Theta(c+v)-\Theta(c)=\int_\Omega D(u)\nabla c\cdot\nabla v+\frac{1}{2}\int_\Omega D(u)\nabla v\cdot\nabla v.$$

Therefore, since $\|D(u)\|_\infty\leq K$ for some positive constant,

$$\int_\Omega D(u)\nabla v\cdot\nabla v = O(\|v\|^2_{H^1(\Omega)}),$$

where here $f(\alpha)=O(\alpha^n)$ means that $\lim_{\alpha\to 0}\frac{f(\alpha)}{\alpha^n}=0$, for some $n\in\mathbb{N}$.

We have shown that Θ is Fréchet differentiable and

$$\langle\Theta'(c),v\rangle=\int_\Omega D(u)\nabla c\cdot\nabla v.$$

We conclude that $\partial\Theta(c)=\{\Theta'(c)\}$ for each $c\in Dom(A)$. Now applying Theorem 5.1.7 [2] and since Θ is monotone, one has that this operator is maximal monotone. This complete the proof of the claim.

To complete the proof, we proceed as follows: first of all we find the solution, (p,u), of (1) with the boundary conditions (4), which existence and uniqueness are guaranteed in [4, 5, 8, 9]. Getting Darcy's velocity, u, from the solution of this problem, we solve (2) with the boundary conditions (5), and using Claim 4.1, Claim 4.2 together with Theorem 3.1 in [3], we obtain the third element of our solution.

To proof the uniqueness of the pair (u,c) for any value of the viscosity, let $(\hat{p},\hat{u},\hat{c})$ be a pair that solves $(L_{\hat{\mu}})$. Consider the pressure $p_1=\frac{\mu}{\hat{\mu}}p$. We have that

$$q=\mathbf{div}(u)=\mathbf{div}\left(\frac{-\mathcal{K}}{\mu}\nabla p\right)=\mathbf{div}\left(\frac{-\mathcal{K}}{\hat{\mu}}\frac{\hat{\mu}\nabla p}{\mu}\right)=\mathbf{div}\left(\frac{-\mathcal{K}}{\hat{\mu}}\nabla p_1\right).$$

This showed that the pair (p_1, u) is a weak solution of the elliptic boundary problem

$$\vartheta = \left(\frac{-\mathcal{K}\nabla\rho}{\hat{\mu}}\right);\ q = \mathbf{div}(\vartheta),\ \rho = 0 \text{ in } \Gamma_D,\ \nabla\rho \cdot n = 0 \text{ in } \Gamma_N,$$

which solution is unique in V and is represented by the pair $(\hat{p}, \hat{u})$. Therefore, we should have $u = \hat{u}$. Now, using the uniqueness of the solution of the parabolic-hyperbolic equation for a given velocity $u = \hat{u}$, we conclude the proof of the assertion and the theorem. ■

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