ISSN 1984-820X
Anais do CNMAC
v.3
PUBLICAÇÃO DA SBMAC
CNMAC2010
XXXIII CONGRESSO NACIONAL DE MATEMÁTICA APLICADA E COMPUTACIONAL

A new algorithm to quantum state discrimination based on semidefinite programming and on the solution of linear systems

Tatiane Evangelista
UFTM - Engineering Departament
38025-180, Uberaba, MG
E-mail: tatilista@gmail.com

Abstract: *The problem of determining the state of a quantum system is a central task in any quantum information processing [2, 6]. However, there are limitations imposed by quantum mechanics on the possibilities to determine the state of a quantum system [3, 4]. For example, nonorthogonal states cannot be discriminated perfectly.*

Eldar [1] showed that the optimal measurement can be formulated as a semidefinite programming problem [7] and that it can be computed in polynomial time. Based on this formulation, Rabelo et al. [5] proposed a method to derive the transformation which maps N nonorthogonal states in a set of states that can be discriminated by usual projective measurements in an extended Hilbert space.

In this work, we propose a new method to obtain the probabilities of inconclusive results in order to simplify the calculation and improve the computational results. The method was written in Matlab and successfully applied to problems with varying from $N = 2$ to $N = 100$. In [1] and [5], their methods were applied for $N = 3$ and $N = 4$, respectively.

Keywords: *quantum discrimination, semidefinite programming, linear systems.*

1 THE PROBLEM

Consider a quantum system prepared in a pure quantum state drawn from a set of given states $\{|Q_i\rangle : i = 1, 2, \ldots, N\}$ in a $N-$dimensional complex Hilbert space. The measurements are constructed in such a way that either the state is correctly detected or the measurement returns an inconclusive result, where the measurement operators are given by

$$\Pi_i = p_i |\tilde{Q}_i\rangle\langle\tilde{Q}_i| \tag{1}$$

and the states $|\tilde{Q}_i\rangle$ are defined by

$$\langle\tilde{Q}_i|Q_k\rangle = \delta_{ik}, \quad \text{if} \quad 1 \leq i, k \leq N, \tag{2}$$

where δ_{ik} is the Kronecker delta.

To obtain the conclusive probabilities $\{p_i : i = 1, \ldots, N\}$, we have to solve the following semidefinite programming problem:

$$min_{p \in R^N} \ \{\langle\mu|p\rangle\} \tag{3}$$

$$\text{s.t} \ \begin{cases} I - \sum_{i=1}^{N} p_i \mathcal{C}_i \geq 0 \\ p_i \geq 0, \end{cases} \tag{4}$$

where $|\mu\rangle$ it is a vector with negative components defined by the priori probabilities of the states $\{|Q_i\rangle\}$ and $\mathcal{C}_i = |\tilde{Q}_i\rangle\langle\tilde{Q}_i|$.

ISSN 1984-820X
Anais do CNMAC
v.3
PUBLICAÇÃO DA SBMAC

2 THE NEW METHOD

In [5], the problem of discrimination of unambiguous states is approached via Neumark's theorem [4], in which the main objective was to obtain the transformation which maps N nonorthogonal pure states in a set of states that can be discriminated by usual projective measurements in an extended Hilbert space. They used a computational procedure that takes as input the ensemble of nonorthogonal states and outputs the best set of discriminable states.

Based on these ideas, we developed a new algorithm for the quantum state discrimination, called quantum discriminator algorithm (QDA). The QDA obtains a set of optimal POVM's, based on the semidefinite programming and the solution of linear systems. The QDA replaces the calculation of the roots of a polynomial of degree 8, as it is done in [5], by the solution of simple linear systems. Furthermore, we developed the algorithm with just one condition on the entry states: the conservation of scalar product.

Now, we will explain in detail every step of QDA.

We start out by rewriting the N entry states to be discriminated in a ladder form in the orthonormal basis $\{|i\rangle, \quad i = 1, \ldots, N\}$:

$$\begin{aligned}
&|Q_1^{lad}\rangle = |1\rangle, \\
&|Q_2^{lad}\rangle = c_{21}|1\rangle + c_{22}|2\rangle, \\
&|Q_3^{lad}\rangle = c_{31}|1\rangle + c_{32}|2\rangle + c_{33}|3\rangle, \\
&\ldots \\
&|Q_N^{lad}\rangle = c_{N1}|1\rangle + c_{N2}|2\rangle + c_{N3}|3\rangle + \ldots + c_{NN}|N\rangle,
\end{aligned} \tag{5}$$

where $\{|Q_i^{lad}\rangle\}$ belong to the Hilbert space of original size N and the coefficients c_{ij} are obtained using the preservation of the scalar product of the entry states, that is, $\langle Q_i^{lad}|Q_j^{lad}\rangle = \langle Q_i|Q_j\rangle$, for all $i, j = 1, \ldots, N$.

In order to obtain the final configuration, we have to solve the system

$$\begin{aligned}
|Q_{f1}\rangle &= g_{11}|1\rangle + g_{1,N+1}|N+1\rangle + \ldots + g_{1,2N-2}|2N-2\rangle + g_{1,2N-1}|2N-1\rangle \\
|Q_{f2}\rangle &= g_{22}|2\rangle + g_{2,N+1}|N+1\rangle + \ldots + g_{2,2N-2}|2N-2\rangle + g_{2,2N-1}|2N-1\rangle \\
|Q_{3f}\rangle &= g_{33}|3\rangle + g_{3,N+1}|N+1\rangle + \ldots + g_{3,2N-2}|2N-2\rangle \\
&\vdots \\
|Q_{fi}\rangle &= g_i|i\rangle + g_{i,N+1}|N+1\rangle + \ldots + g_{i,2N+1-i}|2N+1-i\rangle \\
&\vdots \\
|Q_{fN}\rangle &= g_{NN}|N\rangle + g_{N,N+1}|N+1\rangle, \quad \text{for} \quad i = 3, \ldots, N,
\end{aligned} \tag{6}$$

where $g_{ii} = \sqrt{p_i}$, for $i = 1, \ldots, N$ ($\{p_i\}$ are the solution of the semidefinite programming problem defined above) and the other coefficients g_{ij}, for $i = 1, \ldots, N$ and $j = N+1, \ldots, 2N-1$, are obtained by solving the linear systems

$$\begin{cases}
g_{11}^2 + b^t b = 1 \\
Bb = s \\
\tilde{D}\tilde{D}^t + BB^t = S^t S + ss^t,
\end{cases} \tag{7}$$

where

$$S = \begin{pmatrix} c_{22} & c_{32} & \ldots & c_{N2} \\ 0 & c_{33} & \ldots & c_{N3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_{NN} \end{pmatrix}, s = \begin{pmatrix} c_{21} \\ c_{31} \\ \vdots \\ c_{N1} \end{pmatrix},$$

$$\tilde{D} = \begin{pmatrix} g_{22} & 0 & \ldots & 0 \\ 0 & g_{33} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & g_{NN} \end{pmatrix}, b = \begin{pmatrix} g_{1,N+1} \\ g_{1,N+2} \\ \vdots \\ g_{1,2N-1} \end{pmatrix}$$

ISSN 1984-820X
Anais do CNMAC
v.3
PUBLICAÇÃO DA SBMAC
CNMAC2010
XXXIII CONGRESSO NACIONAL DE MATEMÁTICA APLICADA E COMPUTACIONAL

$$\text{and} \quad B^t = \begin{pmatrix} g_{2,N+1} & g_{3,N+1} & \cdots & g_{N-1,N+1} & g_{N,N+1} \\ g_{2,N+2} & g_{3,N+2} & \cdots & g_{N-1,N+2} & 0 \\ \vdots & \vdots & \ldots & 0 & 0 \\ g_{2,2N-1} & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

While g_{11} looks like a quadratic term, it is a known number obtained by solving the semidefinite programming problem.

This new method to obtain the g_{ij}, for $i = 1, \ldots, N$ and $j = N+1, \ldots, 2N-1$, is much more simpler than the method proposed in [5] and can be used to solve larger problems.

When we arrive at the final discriminable configuration (6), we are ready to make a measurement. If the state collapses to one of the elements of the basis $\{|1\rangle, \ldots, |N\rangle\}$, we have a perfect discrimination. Otherwise, if the state collapses to one of the elements of the basis $\{|N+1\rangle, \ldots, |2N-1\rangle\}$, we cannot say which state was detected. The figure below illustrates this procedure.

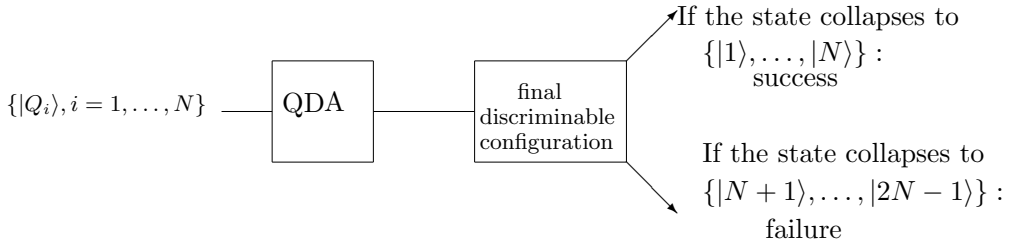

Figure 1: Measurement procedure for N nonorthogonal quantum states via QDA.

3 Computational results

First, we will see a simple example for the application of QDA, for $N = 3$. Consider the ensemble $\{(|Q_1\rangle, 0.6); (|Q_2\rangle, 0.2); (|Q_3\rangle, 0.2)\}$, where

$$\begin{cases} |Q_1\rangle = |1\rangle, \\ |Q_2\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{3}}|3\rangle, \\ |Q_3\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle - \frac{1}{\sqrt{3}}|3\rangle. \end{cases} \tag{8}$$

Applying the QDA, we have the following steps:

- We start out by rewriting the $N = 3$ entry states to be discriminated in a ladder form in the orthonormal basis $\{|i\rangle, \quad i = 1, 2, 3\}$:

$$\begin{cases} |Q_1^{lad}\rangle = |1\rangle \\ |Q_2^{lad}\rangle = 0.58|1\rangle + 0.82|2\rangle \\ |Q_3^{lad}\rangle = 0.58|1\rangle + 0.82|3\rangle. \end{cases} \tag{9}$$

- The solution of the problem (3) can be calculated using the Matlab software package through the CSDP. The symbol CSDP can have several different meanings. In this text, we are referring to the package written by Brian Borchers on C for solving semidefinite programming problems [8]. This package includes an executable program for solving the

problems, a library that provides routines in C and functions to use the application in Matlab, we obtain

$$|p\rangle = \begin{pmatrix} 0.33 \\ 0.33 \\ 0.33 \end{pmatrix}. \tag{10}$$

- Solving the linear system (7) obtain coefficients g_{ij}, for $i = 1, 2, 3$ for $j = 4, 5$ and the $g_{ii} = \sqrt{p_i}$, where p_i are solution of the problem semidefinite programming, for $i = 1, 2, 3$. So obtain the final configuration

$$\begin{cases} |Q_{f1}\rangle = 0.58|1\rangle + 0.71|4\rangle + 0.41|5\rangle, \\ |Q_{f2}\rangle = 0.58|2\rangle + 0.41|4\rangle + 0.71|5\rangle, \\ |Q_{f3}\rangle = 0.58|3\rangle + 0.82|4\rangle. \end{cases} \tag{11}$$

In the Table 1, we present the medium time for each N (considering the 10 random instances) necessary for the QDA to obtain the final discriminable configuration for both packages (the code was written in Matlab 7.0.1 and all the experiments were carried out on an Intel Core 2, 1.66 GHz and 1GB RAM, running Windows XP).

N	CSDP
2	1.4344s
3	1.7156s
4	1.7706s
5	2.1174s
6	2.0282s
7	2.2936s
8	2.9421s
9	2.4860s
10	2.0514s
20	2.6954s
30	4.3342s
40	6.1859s
50	9.0860s
60	14.8456s
70	20.8436s
80	32.8407s
90	44.4249s
100	1min 11.025s

Table 1: Medium time for each $N = 2, 3, \ldots, 10, 20, 30, \ldots, 100$.

4 Conclusion

We proposed a new algorithm for quantum state discrimination, based on semidefinite programming and on the solution of linear systems. The algorithm obtains the best set of POVM measurements for the problem, through the extension of the Hilbert space of N to $2N - 1$ dimensions, where N is the number of states to be discriminated. The algorithm was written in Matlab and successfully applied on problems with dimensions varying from $N = 2$ to $N = 100$.

References

[1] Y.C. Eldar, A semidefinite programming approach to optimal unambiguous discrimination of quantum states, *IEEE Transactions on Information Theory, 49, (2003) 446-456.*

ISSN 1984-820X
Anais do CNMAC
v.3
PUBLICAÇÃO DA SBMAC
CNMAC2010
XXXIII CONGRESSO NACIONAL DE MATEMÁTICA APLICADA E COMPUTACIONAL

[2] C. Fuchs, "Distinguishability and Accessible information in Quantum Theory", Ph.D. Thesis, University of New Mexico, 1995.

[3] M.A. Nielsen and I.L. Chuang, "Quantum Computation and Quantum Information", Cambridge University Press, 2000.

[4] A. Peres, "Quantum Theory: Concepts and Methods", Kluwer, Boston, 1993.

[5] W. R. M. Rabelo, A. G. Rodrigues and R. O. Vianna, An algorithm to perform POVMs through Neumark theorem: application to the discrimination of non-orthogonal pure quantum states, International Journal of Modern Physical C, 17, (2006) 1203, 1-16.

[6] L. Roa, J. C. Retamal and C. Saavedra, Quantum-state discrimination, *Physical Review A, 66, (2002) 12103, 1-4.*

[7] H. Wolkowicz, R. Saigal and L. Vandenbergue, "Handbook of Semidefinite Programming: Theory, Algorithms and Applications", Kluwer, 2000.

[8] B. Borchers, Optimization methods and software, 11, (1999), 613-623.