

# Group and graph theories applied to the analysis of mechanisms and parallel robots

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**Abstract:** *This paper addresses applications of group and graph theories tools to simplify the analysis of kinematic chains associated with mechanisms and parallel robots. For the purpose of this analysis, a kinematic chain is described by its properties, i.e. degrees-of-control matrix, connectivity matrix and redundancy matrix. Kinematic chains are represented by graphs and thus, the symmetry of a kinematic chain is the same as the symmetry of its graph. The symmetry group of the graph is associated with the graph symmetry. By using the group structure induced by the symmetry we prove that degrees-of-control, connectivity and redundancy are invariants by the action of the automorphism group of the graph.*

**Keywords:** *kinematic chains, mechanisms, parallel robots, graph symmetry, automorphism group, actions, orbits.*

## 1 Introduction

Mathematical models are commonly difficult to handle in a general setting. Symmetry in mathematical models is useful to simplify the understanding of a model and to determine the patterns for which the model is appropriate. Thus, it is a common strategy to study cases of symmetry in order to learn more about a model. In our setting, the mathematical model associates a kinematic chain with a graph. Belfiore and Di Benedetto [1], Liberati and Belfiore [9] and Martins and Carboni [10] discuss how the topological structure of a kinematic chain of a parallel manipulator can be described quite extensively by degrees-of-control, connectivity and redundancy matrices. These matrices are square symmetric with dimension  $n \times n$ , where  $n$  is the number of links of the kinematic chain. Our aim is to apply group and graph theories to reduce the size of the degrees-of-control matrix, connectivity matrix and redundancy matrix of a kinematic chain. In this context, the graph symmetry plays an important role because it provides a group structure.

In order to achieve our aims, we investigate symmetries and invariants by the action of the automorphism group of the graph representing kinematic chains and parallel robots. Symmetries of graphs are related to automorphisms [2, 13]. By exploring these symmetries it is possible to reduce the matricial representation of properties important to the kinematic analysis of kinematic chains. It is shown that the matrix size is reduced from  $n \times n$  to  $o \times n$ , where  $n$  is the number of links and  $o$  is the number of orbits by the action of the automorphism group of the graph. Higher graph symmetry means a smaller number of orbits  $o$ , as will be clearly shown through examples. The main result of this study was to prove that the connectivity matrix is invariant by the action of the automorphism group of the graph, the invariance is the main tool used to reduce the size of the matrices.

Group theory has been used by some authors in the context of analyzing kinematic chains of mechanisms and parallel robots. Tsai [19] uses the symmetry group of a kinematic chain to identify when two kinematic chains are identical (isomorphism problem). Simoni *et al.* [16, 15] have applied group theory tools in the enumeration of kinematic chains, mechanisms and parallel robots.

## 2 Graphs and Symmetry

A graph  $X = (V, E)$  consists of a finite set  $V = V(X)$  of vertices and a family  $E = E(X)$  of subsets of  $V$  of size two called edges. Usually, the pair  $\{x, y\}$  denotes an edge, and the number of edges incident to a vertex  $v$  is the degree of the vertex  $v$  ( $\deg(v)$ ). A subgraph of a graph  $X$  is a graph  $Y$  such that  $V(Y) \subseteq V(X)$ ,  $E(Y) \subseteq E(X)$ . The distance between two vertices  $x$  and  $y$ , denoted by  $\delta(x, y)$ , is the length of the shortest path between  $x$  and  $y$ . It is important to note that a kinematic chains of mechanisms and parallel robots can be uniquely represented by a graph whose vertices correspond to the links of the chain and whose edges correspond to the joints of the chain [19, 12]. Figure 1 shows this correspondence, Figure 1(a) shows the classical Stephenson kinematic chain with labeled links and Figure 1(b) shows the corresponding graph. From now on, the term graph will be used to mean a kinematic chain, unless otherwise stated.

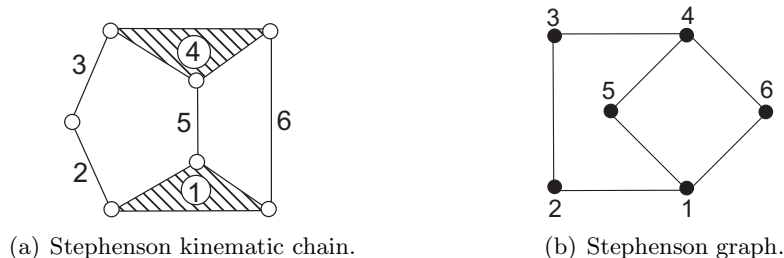


Figure 1: Biunivocal correspondence between graphs and kinematic chains.

### 2.1 Isomorphisms and Automorphisms

Given a graph  $X$ , a bijective map  $\sigma : V(X) \rightarrow V(X)$  defines a permutation of the elements of  $V(X)$ . Assuming  $V(X)$  has  $n$  elements, the set of permutations endowed with the operation of composition is the group  $S_n$ .

**Definition 1 (Isomorphisms and automorphisms of graphs)** *Two graphs  $X$  and  $Y$  are isomorphic if there is a bijection  $\sigma : V(X) \rightarrow V(Y)$  such that  $\{x, y\} \in E(X) \Leftrightarrow \{\sigma(x), \sigma(y)\} \in E(Y)$ . If isomorphism exists between two graphs, then the graphs are called isomorphic and we write  $X \simeq Y$ . The automorphism of a graph is the graph's isomorphism with itself. The automorphism group of a graph  $X$  is denoted by  $\text{Aut}(X)$ .*

Herein, we use some results for invariants of isomorphism and automorphism groups of graphs found in the literature [4, 17]. These results are important to prove Theorems 9 and 10 in Section 4 and are summarized below:

**Remark 1** *Let  $X$  be a graph,  $Y$  a subgraph of  $X$  and  $\sigma$  an element of  $\text{Aut}(X)$ .*

- 1 - **Degree invariance:**  $\deg(\sigma(x)) = \deg(x)$ , for all  $x \in V(X)$ ;
- 2 - **Distance invariance:**  $\delta(\sigma(x), \sigma(y)) = \delta(x, y)$ , for all  $x, y \in V(X)$ ;
- 3 - **Subgraph invariance:**  $\sigma(Y) \simeq Y$ , i.e. they are isomorphic.

Proofs of these invariant remarks are found in [4, 17].

## 2.2 Orbits

The orbit of a graph vertex corresponds to the subset of vertices for which the vertex is moved by the action of the automorphism group of the graph. Let us consider an example.

**Example 1** Let  $X$  be the Stephenson graph shown in Figure 1(b) (pg. 2). In this case,  $Aut(X) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ , where  $\sigma_1 = (1)(2)(3)(4)(5)(6)$ ,  $\sigma_2 = (1)(2)(3)(4)(56)$ ,  $\sigma_3 = (14)(23)(5)(6)$  and  $\sigma_4 = (14)(23)(56)$ . The generator set is  $Aut(X) = \langle \sigma_2, \sigma_3 \rangle$ . The action of the automorphism group in the Stephenson graph is shown in Figures 2(a), 2(b), 2(c) and 2(d).

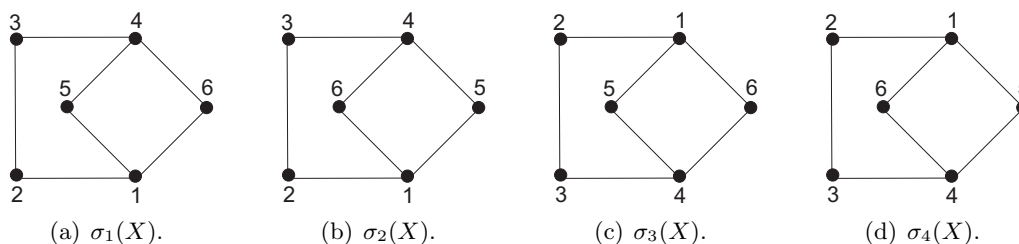


Figure 2: Action of the automorphism group in the Stephenson graph.

The orbits are:

$$\mathcal{O} = \{\{1, 4\}, \{2, 3\}, \{5, 6\}\} = \{\mathcal{O}_{III}, \mathcal{O}_{II}, \mathcal{O}_I\}$$

where  $\mathcal{O}_{III} = \{1, 4\}$ ,  $\mathcal{O}_{II} = \{2, 3\}$  and  $\mathcal{O}_I = \{5, 6\}$ . The Stephenson graph has three different orbits resulting in classical Stephenson I ( $\mathcal{O}_I$ ), II ( $\mathcal{O}_{II}$ ) and III ( $\mathcal{O}_{III}$ ) mechanisms fixing a representative link of each orbit (for more details consult [16, 15]).

## 2.3 Symmetry in kinematic chains

Since our aim is to apply symmetry to simplify the kinematic chain analysis, and kinematic chains are represented in biunivocal correspondence by graphs, it is necessary to define what is meant by graph symmetry.

The symmetry of a graph corresponds to an element of the automorphism group of the graph. According to Erdős and Rényi [2], a graph is considered to be symmetric when it has more than one automorphism, i.e. the automorphism group has a degree greater than 1. In the definition below we extend the concept of graph symmetry to kinematic chains.

**Definition 2 (Symmetry of a kinematic chain)** The symmetry of a kinematic chain is the symmetry of its corresponding graph. A kinematic chain is symmetric when it has more than one automorphism.

## 3 Properties of Kinematic Chains

In this section, some fundamental properties of kinematic chains are introduced. These are essential for topological analysis and number synthesis of mechanisms and parallel robots.

**Definition 3 (Mobility)** The number of degrees of freedom, or mobility ( $M$ ), of a kinematic chain is the number of independent parameters required to completely specify the configuration of the kinematic chain in space, with respect to one link chosen as the reference.

The mobility of a kinematic chain, with  $n$  links and  $g$  single degree of freedom joints, may be calculated by the general mobility criterion [12, 6] applied to a set of  $n$  links and  $g$  single degree of freedom joints:

$$M = \lambda(n - g - 1) + g \quad (1)$$

where  $\lambda$  is the order of the screw system to which all the joint screws belong.

Using the graph representation of a kinematic chain (see Fig. 1, pg. 2), the general mobility criterion is given by

$$M = \lambda (|V| - |E| - 1) + |E| \quad (2)$$

where  $|V|$  is the number of graph vertices (*i.e.* links) and  $|E|$  is the number of graph edges (*i.e.* joints)[19, 12].

The *connectivity*  $C_{ij}$  between two links  $i$  and  $j$  of a kinematic chain is the relative mobility between links  $i$  and  $j$  [6]. Different algorithms for connectivity calculations have been proposed by Shoham and Roth [14], Belfiore and Di Benedetto [1], Liberati and Belfiore [9] and Martins and Carboni [10].

**Definition 4 (Connectivity [10])** In a kinematic chain represented by a graph  $X$ , the connectivity between two links  $i$  and  $j$  is

$$C_{ij}(X) = \min\{D[i, j], M'_{\min}, \lambda\} \quad (3)$$

where  $D[i, j]$  is the distance between vertices  $i$  and  $j$  of  $X$ ,  $M'_{\min}$  is the minimum mobility of any closed-loop biconnected subgraph of  $X$  containing vertices  $i$  and  $j$ , and  $\lambda$  is the order of the screw system.

**Definition 5 (Degrees-of-control [10])** In a kinematic chain represented by a graph  $X$ , the degrees-of-control between two links  $i$  and  $j$  is

$$K_{ij}(X) = \min\{D[i, j], M'_{\min}\} \quad (4)$$

Based on the definition of degrees-of-control and connectivity, the definition of redundancy can be introduced.

**Definition 6 (Redundancy [1])** In a kinematic chain represented by a graph  $X$ , the redundancy between two links  $i$  and  $j$  is the difference between  $K_{ij}(X)$  and  $C_{ij}(X)$

$$R_{ij}(X) = K_{ij}(X) - C_{ij}(X) \quad (5)$$

The importance of the connectivity, degrees-of-control and redundancy is emphasized by several authors [1, 6, 18].

**Example 2** Let's consider the planar kinematic chain  $X$  shown in Fig. 3 where the mobility is  $M = 3$ . The adjacency matrix  $A(X)$  is given by Equation 6 and the connectivity  $C(X)$  is given by Equation 7.

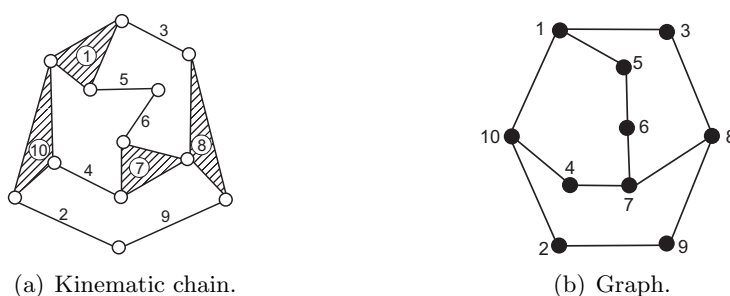


Figure 3: Kinematic chain  $X$  and graph representation.

$$A(X) = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix} \quad (6) \quad C(X) = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{matrix} & \begin{bmatrix} 0 & 2 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 1 \\ 2 & 0 & 3 & 2 & 3 & 3 & 3 & 2 & 1 & 1 \\ 1 & 3 & 0 & 3 & 2 & 3 & 2 & 1 & 2 & 2 \\ 2 & 2 & 3 & 0 & 3 & 2 & 1 & 2 & 3 & 1 \\ 1 & 3 & 2 & 3 & 0 & 1 & 2 & 3 & 3 & 2 \\ 2 & 3 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 3 \\ 3 & 3 & 2 & 1 & 2 & 1 & 0 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 & 3 & 2 & 1 & 0 & 1 & 3 \\ 3 & 1 & 2 & 3 & 3 & 3 & 2 & 1 & 0 & 2 \\ 1 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 2 & 0 \end{bmatrix} \end{matrix} \quad (7)$$

In this case, the degrees-of-control matrix  $K(X)$  is equal to the connectivity matrix  $C(X)$ ,  $K(X) = C(X)$ , and the redundancy matrix is null,  $R(X) = 0$ .

## 4 Theorems and Applications

This section considers the applications of the group and graph theories to analysis of kinematic chains. First, we prove the invariance of degrees-of-control, connectivity and redundancy of kinematic chains by the action of its automorphism group of the graph.

**Lemma 7 (Mobility invariance)** *The mobility  $M$  of a graph (kinematic chain) is invariant by the action of the automorphism group of the graph.*

**Proof:** The proof follows from Definition 1. An automorphism of a graph is an isomorphism with itself and thus, the graph structure is preserved. As we can see in Example 1, the automorphism group of the graph results in the relabeling of the graph vertices and consequently the number of vertices  $|V|$ , the number of edges  $|E|$  and the order of the screw system  $\lambda$  remain the same. Consequently, the mobility, Equation 2, is invariant.

**Lemma 8 (Subgraph mobility invariance)** *The mobility  $M$  of a subgraph (subchain) is invariant by the action of the automorphism group of the graph.*

**Proof:** The proof follows from Remark 1 and Lemma 7. Remark 1 proves that a subgraph is invariant by the action of its automorphism group and thus, the structure of the subgraph ( $|V|$ ,  $|E|$ ,  $\lambda$ ) remains the same. Lemma 7 proves that the mobility is invariant. Consequently, the subgraph mobility is invariant.

**Theorem 9 (Degrees-of-control invariance)** *Let  $X$  be a graph (kinematic chain) and  $Aut(X)$  its automorphism group. The degrees-of-control matrix  $K(X)$  of the kinematic chain is invariant by the action of the automorphism group of the graph.*

**Proof:** The degrees-of-control is given by  $K_{ij} = \min\{D[i, j], M'_{\min}\}$ , see Equation 4. To prove this theorem, it is necessary to show that  $D$  matrix and  $M'_{\min}$  are invariant by the action of the automorphism group. According to Remark 1 the distance of any pair of vertices is invariant by the action of the automorphism group of the graph, i.e.  $D[i, j] = D[\sigma(i), \sigma(j)]$ . Therefore, the  $D$  matrix is invariant by the action of the automorphism group of the graph. According to Remark 1 any subgraph is invariant by the action of the automorphism group of the graph, therefore,  $M'_{\min}$  is also invariant.

**Theorem 10 (Connectivity invariance)** *Let  $X$  be a graph (kinematic chain) and  $Aut(X)$  its automorphism group. The connectivity matrix  $C(X)$  of the kinematic chain is invariant by the action of the automorphism group of the graph.*

**Proof:** The proof follows from Theorem 9. The connectivity is given by  $C_{ij} = \min\{K_{ij}, \lambda\}$ , see Equation 3.  $K_{ij}$  is invariant according to Theorem 9 and  $\lambda$  is a property of the kinematic chain (it is not dependent on the graph) and therefore it is constant.

**Corollary 11 (Redundancy invariance)** *Let  $X$  be a graph (kinematic chain) and  $Aut(X)$  its automorphism group. The redundancy matrix  $R(X)$  of the kinematic chain is invariant by the action of the automorphism group of the graph.*

**Proof:** The proof follows straightforwardly from Theorems 9 and 10. The redundancy is given by  $R_{ij}(X) = K_{ij}(X) - C_{ij}(X)$  (see Equation 6).  $K_{ij}(X)$  and  $C_{ij}(X)$  are invariants according to Theorems 9 and 10, consequently,  $R_{ij}(X)$  is invariant.



Theorems 9 and 10 and Corollary 11 state that the connectivity, degrees-of-control and redundancy are symmetric properties of a kinematic chain, *i.e.* elements which are symmetric by the action of the automorphism group of the graph have the same properties. Considering that symmetric links are identified by the orbits of the automorphism group of the graph, it is possible to reduce the matricial representation considering one representative element of each orbit.

#### 4.1 Application to Planar Parallel Mechanisms

Let  $X$  be the kinematic chain of the planar parallel mechanism shown in Fig. 3. In this case,  $Aut(X) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ , where  $\sigma_1 = (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)$ ,  $\sigma_2 = (1\ 7)(2\ 9)(3\ 4)(5\ 6)(8\ 10)$ ,  $\sigma_3 = (1\ 8)(2\ 6)(5\ 9)(7\ 10)$  and  $\sigma_4 = (1\ 10)(2\ 5)(3\ 4)(6\ 9)(7\ 8)$ . The generator set is  $Aut(X) = \langle \sigma_2, \sigma_3 \rangle$ . The automorphism group of the graph can be obtained with the *nauty* [11]. The *nauty* (No AUTomorphisms, Yes?) is a set of very efficient C language procedures for determining the automorphism group of a graph with colored vertices. It provides this information in the form of a set of generators, the size of the group, and the orbits of the group.

The orbits are:

$$\mathcal{O} = \{\{1\ 7\ 8\ 10\}; \{2\ 5\ 6\ 9\}; \{3\ 4\}\} = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}$$

where  $\mathcal{O}_1 = \{1\ 7\ 8\ 10\}$ ,  $\mathcal{O}_2 = \{2\ 5\ 6\ 9\}$  and  $\mathcal{O}_3 = \{3\ 4\}$ . The matrices presented in Equations 6 and 7 are reduced to matrices presented in Equation 8 and 9. In this representation we chose as representative elements of each orbit the elements (links) 1, 2 and 3.

$$A_r(X) = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} \\ \begin{matrix} \mathcal{O}_1 \\ \mathcal{O}_2 \\ \mathcal{O}_3 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{matrix} \quad (8) \quad C_r(X) = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} \\ \begin{matrix} \mathcal{O}_1 \\ \mathcal{O}_2 \\ \mathcal{O}_3 \end{matrix} & \begin{bmatrix} 0 & 2 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 1 \\ 2 & 0 & 3 & 2 & 3 & 3 & 3 & 2 & 1 & 1 \\ 1 & 3 & 0 & 3 & 2 & 3 & 2 & 1 & 2 & 2 \end{bmatrix} \end{matrix} \quad (9)$$

Note that the matrices are reduced from  $10 \times 10$  to  $3 \times 10$ . Note also that the matrix representations  $A_r(X)$  and  $C_r(X)$  are more compact than the original  $A(X)$  and  $C(X)$ . The more symmetric the kinematic chain the smaller is its representation. As most parallel mechanisms found in the literature are symmetric, this representation is particularly advantageous.

## 5 Conclusions

The main contribution of this paper is to prove the invariance of connectivity, degrees-of-control and redundancy by the action of the automorphism group of the graph. The connectivity, degrees-of-control and redundancy are symmetric properties of a kinematic chain, *i.e.* links which are symmetric by the action of the automorphism group of the graph have the same properties. Considering that symmetric links are identified by the orbits of the automorphism group of the graph, we reduce the matricial representation considering one representative element of each orbit. Thus, the order of the matrices are reduced from  $n \times n$  to  $o \times n$  where  $n$  is the number of links of the kinematic chain and  $o$  is the number of orbits of the automorphism group of the graph.

Considering that the majority of parallel robots in the literature have symmetric kinematic chains [3, 7, 8, 5] the reduced representation offers considerable advantages. As shown in the examples, if a kinematic chain has symmetry it is possible to obtain a gain in terms of the storage of matrices, and in the simplicity of the kinematic analysis. These techniques can also be applied to kinematic chains of serial and hybrid robots. The only cases for which the theory presented herein is not advantageous is when the graph is fully asymmetric, *i.e.* in rare practical cases.

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