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# Exact solutions of Bratu and Liouville equations

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**Abstract:** *The solution of the Bratu equation used in combustion, and the Liouville equation used in quantum mechanics and field theory, is shown to reduce to the solution of the equation* $v_x = v^2/2 + f(x)$.

**Keywords :** Non linear differential equations, general solution, combustion.

## I Introduction

Liouville solved in 1850 the second order PDE $u_{xu} + \lambda Exp[u] = 0$, which is now associated with the so-called Liouville model in quantum field theory. Bratu solved in 1914 the somewhat similar second order ODE $u_{xx} + \lambda Exp[u] = 0$, which is a basic 1-D model in combustion theory. We show that the study of these equations can be reduced to the nonlinear first-order ODE $v_x = v^2 + f$.

For simplicity, we shall only assume the *local* theory of differential equations, which guarantees the existence (and, under stricter conditions, the uniqueness) of a solution for an initial value problem in a small neighborhood of the initial point. On the other hand, we shall always be precise as to the smoothness of functions defined on the given region, in terms of the $C^k$ classes of continuous functions with $k$ continuous derivatives.

We use subscripts for ordinary derivatives of ODEs, as well as for partial derivatives of PDEs.

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# II The basic equation $v_x = v^2/2 + f$.

We wish to find $C^1$ solutions $v(x)$ of the equation $v_x = v^2/2 + f(x)$ for a given function $f(x)$. We shall refer to this equation as *the basic equation.*

We recall the general theory of initial value problem of the general form $v_x = F(v, x)$, $v(a) = v_0$. Existence of a local solution is guaranteed if $F$ is continuous in $x$ and uniformly continuous in $v$, and its uniqueness is then guaranteed if $F$ is uniformly Lipschitz in $v$. Thus, a unique local solution of $v_x = v^2/2 + f$ is guaranteed if $f$ is continuous in $x$.

As an example, we consider the basic equation when $f(x) \equiv k$ is a constant. Some special solutions of this type are

$$k = 0, v(x) = -2/x; \qquad k = 2, v(x) = 2tan(x);$$

$$k = -2, v(x) \in \{-2tanh(x), -2coth(x), 2, -2\}.$$

These solutions can be derived easily by separation of variables (see e.g. [5]), and some of them have singular points, showing their local nature.

When $f(x)$ is not a constant, it is generally impossible to find a closed form solution, and numerical methods should be used. For examples, initial value problems associated with this equation can be solved using Eulers method, or more generally, Runge-Kutta methods.

We mention that if $f(x)$ is $C^k$ then "by bootstrapping" the solution $v(x)$ is $C^{k+1}$. This follows by repeated differentiation of the ODE. Thus, for example, solutions of $v_x = v^2/2 + k$ are $C^\infty$.

In this presentation we shall derive the solution of two classical models used in the applications by reduction to the basic equation. First, however, we consider the rather trivial ODE $u_x + \lambda Exp[u] = 0$ with $\lambda \in \mathbb{R}$ fixed. Differentiating once, we get $0 = u_{xx} + e^{\lambda u} u_x = u_{xx} - u_x^2$. Thus, $v := 2u_x$ solves the basic equation $v_x = v^2/2$, whose solution is specified above. Thus, the solution of the ODE $u_x + Exp[\lambda u] = 0$ can be reduced to the simple system

$$u = ln[-v/2\lambda], \qquad v_x = v^2/2.$$

The applications to follow may be regarded as non-trivial extensions of this simple example.

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# III The Bratu equation

The Bratu equation in $n$ dimensions is $\Delta u + \lambda Exp[u] = 0$, where $\Delta$ is the Laplacian, see see [11]. This nonlinear PDE, coupled with appropiate boundary conditions, is used to model the temperature distribution in combustion models. In particular, treating combustion in a star, and assuming radial solutions (though this in general is not the case), this PDE is reduced to the ODE $rU_{rr} + (n-1)U_r + r\lambda Exp[u] = 0$. The three-dimensional model was considered by Emden in his investigations on the sun's core temperatures in his 1904 book [9] see also [7], [4] .

Analytic solutions for the radial Bratu equation are known only for $n = 1, 2$. However, it is known that the number of solutions undergoes a bifurcation when $\lambda$ crosses a critical value; the bifurcation picture is known for all $n$ (see e.g. [10], [11], [12], [13], [14], [19]).

Our second application of the basic equation is the Bratu equation in one dimension,

$$u_{xx} + \lambda e^u = 0, \tag{1}$$

solved by Bratu in 1914 [6] and revisited by Gelfand in 1964 [11]. Usually, one seeks a positive solution in $[-1, 1]$ under the Dirichlet conditions $u(-1) = u(1) = 0$. However, we shall consider the *general* solution for the ODE, independent of boundary conditions. We find a hidden connection with the equation $v_x = v^2 + k$, which allows a more elegant analysis of its solutions. We assume the solution to be defined around the origin.

**Lemma 1.** *Let $\lambda \in \mathbb{R}$ be given. The $C^2$ function $u(x)$ is a local solution of (1) at $x = 0$ iff $v = u_x$ is a local solution of $v_x = v^2/2 + k$ for some $k \in \mathbb{R}$. Moreover, if $\lambda \neq 0$ we have*

$$u(x) = ln(-v_x/\lambda). \tag{2}$$

Verification for $\lambda = 0$ is trivial: $u(x)$ is a first order polynomial iff it satisfies $u_{xx} = 0$. So assume $\lambda \neq 0$. We have $v_x = u_{xx} = -\lambda Exp[u]$, or $u = ln[-v_x/\lambda]$ as claimed. Differentiating once more we get $v_{xx} = -\lambda Exp[u]u_x = v_x v = (v^2)_x/2$, and then integration produces $v_x = v^2/2 + k$ as required. The converse direction is similar. □

We comment that although we assumed $u$ to be $C^2$, we have used its third derivative. This is no problem since, by bootstrapping, any $C^2$ solution $u$ is $C^\infty$.

A similar result can be obtained for the radial solutions $U(r)$ 2-dimensional Bratu equation $rU_{rr} + U_r + \lambda r Exp[U] = 0$ at some $r > 0$. A procedure similar to Lemma 1 reduces this second order ODE to the first order ODE $rv_r = v^2/2 + k$ for some $k$, which again can be solved completely by separation of variables.

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# IV The Liouville equation

In 1850 (see page 597 of Monge's book [18]), J. Liouville presented a formula which provides solutions for the second order hyperbolic nonlinear PDE

$$u_{xy} + \lambda e^u = 0. \tag{3}$$

In recent days, variants of this equation have assumed a central role in quantum physics and field theory, where it is known as the Liouville model (see e.g. [8], [15]).

In a 2-page 1853 paper, the same author went on to show that his formula gives the *general* solution for this equation. His proof contains several typos which make it difficult to follow; more seriously, it is incomplete in that it does not address uniqueness considerations for boundary/initial value problems associated with the equation. However, it does point out an important link with the basic equation $v_x = v^2/2 + f$. Below we complete his proof.

**Theorem 1.** *Let $\lambda \in \mathbb{R}$ $(\lambda \neq 0)$ be given. A $C^2$ function $u(x,u)$ near the origin is a solution of (3) iff there exist $C^2$ functions $\phi(x), \psi(y)$ near the origin satisfying both*

$$\begin{array}{cc} (i) & \lambda\phi_x(a)\psi_y(b) > 0, \qquad \phi(a) + \psi(b) \neq 0 \\ (ii) & u = ln\left[2\phi_x\psi_y/\lambda(\phi+\psi)^2\right]. \end{array} \tag{4}$$

One implication in the Theorem is obvious. Assume that the function $u(x,y)$ is of the form (4). Direct differentiation shows that indeed $u$ solves (3). This is best shown by splitting $u = ln(2\phi_x\psi_y/\lambda) - ln(\phi+\psi)^2$, leading to

$$v := u_x = \phi_{xx}/\phi_x - 2\phi_x/(\phi+\psi) \tag{5}$$

and then to $u_{xy} = 2\phi_x\psi_y/(\phi+\psi)^2 = Exp[\lambda u]$ as required. Observe the remarkable fact that $v(x,y)$, a function of $y$, solves an ODE which is independent of $y$. Furthermore, the ODE it solves is no other but the basic equation treated in section 1!

The converse direction is harder: if $u(x,y)$ solves (3), how do we produce functions $\phi(x), \psi(y)$ so that $u$ assumes the form (4)? To answer this quastion, we need to understand the "correct initial value problem" associated with the PDE (3) which would guarantee uniqueness. A plausible formulation, given below, requires initial value data spread over both the x- and y-axes!

**Lmma 2.** *A local $C^2$ solution $u(x,y)$ of (3) around $(0,0)$ is uniquely determined by the following local data for small $x, y$:*

$$f(x) = v_x(x,0) - v^2(x,0)/2 \quad \xi = v(0,0) \quad (v = u_x); \qquad g(y) = u(0,y).$$

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**Proof.** The Fundamental Theorem, plus the definitions of $g, v$, gives $u(x,y) = \int_0^x v(t,y)dt + g(y)$. As $g$ is specified, we need only show that $v$ is uniquely determined by $f, g, \xi$. Indeed, as $v_y = u_{xy} = -\lambda Exp[u]$, we can establish that

$$[v_x - v^2/2]_y = v_{xy} - vv_y = [u_{xy} + \lambda Exp[u]]_x = 0.$$

Thus, there exists $\tilde{f}(x)$ such that $v_x - v^2/2 = \tilde{f}(x)$, independent of $y$. Setting $y = 0$ we get $\tilde{f}(x) = v_x(x,o) - v(x,0)^2/2 = f(x.)$ We conclude that

$$v_x - v^2/2 = f(x) \tag{6}$$

where by the assumptions the right hand side is continuous. As we already know, a unique solution $v(x,y)$ exists (for $y$ fixed) if the initial value at $x = 0$, i.e. the value $w(y) = v(0,y) = u_x(0,y)$, is specified. The function $w(y)$ is $C^1$ and satisfies $w_y = u_{xy}|_{x=0} = -\lambda Exp[u]|_{x=0} = -\lambda Exp[g]$. We can construct $w$ uniquely from the ODE $w_y = -\lambda Exp[g]$ and the initial value $\xi = w(0) = v_(0,0)$. Thus, the unique construction of $u$ has been reduced to the local data specified in the Lemma. □

In particular, if $u(x,y)$ has the form (4) then the local data specified in the Lemma can be calculated by direct differentiation and substitution. Defining the initial values

$$[\eta_0, \eta_1, \eta_2] = [\phi, \phi_x, \phi_{xx}]|_{x=0}, \qquad \nu = \psi|_{y=0}, \tag{7}$$

we find that

$$\xi = \eta_2/\eta_1 - 2\eta_1/(\eta_0 + \nu), \tag{8}$$

$\psi(y)$ solves the quadratic equation

$$\lambda(\Psi^2 + 2(\lambda\eta_0 - \eta_1/E)\Psi + \lambda\eta_0^2 = 0 \qquad (\Psi = \psi(y),\ E = Exp[g(y)]. \tag{9}$$

The discriminant $4\eta_1(\eta_1 - 2E\lambda\eta_0)/E^2$ is nonnegative if

$$\lambda = 0 \text{ and } \eta_1 = 0; \text{ or } |\eta_0| >> 0 \text{ and } \lambda\eta_0\eta_1 > 0.$$

Finally,

$$f(x) = \phi_{xxx}/\phi_x - 3\phi_{xx}^2/2\phi_x^2. \tag{10}$$

Equipped with these relations, we can now prove Theorem 1 by simply reverting the direct analysis developed above.

**Proof of Theorem 1.** Let $u(x,y)$ be a $C^2$ solution of (3). Choose constants $\eta_i, \nu$ which satisfy (IV,8)). Choose $\psi$ by solving for each $y$ the quadratic relation (9) for

$\Psi = \psi(y)$. Define $f(x)$ through the basic equation: $f = v_x = v^2/2 + f, \qquad v = u_x$. Define $\mu(x)$ via the initial value problem

$$\mu_x - \mu^2/2 = f(x), \qquad \mu(0) = \eta_2/\eta_1.$$

Then define $\phi(x)$ through the equation

$$[ln[\phi_x]_x = \mu, \qquad \phi(0) = \eta_0, \quad \phi_x(0) = \eta_1,$$

namely, $\phi(x) = \eta_0 + \eta_1 \int_0^x Exp[\int_0^t \mu(s)\, ds\, dt$. Now form the function

$$\tilde{u} = ln\left[(2\phi_x\psi_y)/(\lambda(\phi+\psi)^2)\right]. \tag{11}$$

As commented before, $\tilde{u}(x,y)$ is a solution of (3). Moreover, by the construction, $\tilde{u}$ and $u$ have the same initial data $f(x), g(y), \xi$. Thus, $\tilde{u} = u$, completing the proof. □

# V Epilogue: the non-radial Bratu equation

In her 1983 book [3], see also [1], Catherine Bandle treated non-radial solutions of the 2-dimensional Bratu equation. She was the first to observe the similarity between the Bratu and Liouville model and to apply Liouville's technique (Bratu had used a different technique in [2]), obtaining a large class of solutions. For solutions of the corresponding boundary value problem in disks and rings see [2], [16].

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